Math 617: Day 1

Measure Theory

(From my perspective) there are a few basic motivations for measure theory:

① To expand the notion of integrability &, in particular, to be able to define the definite integral of functions like the Dirichlet function:

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

② As an outgrowth of the above, to define distributions &, in particular, function spaces like L^p. In turn, these are essential to modern approaches to partial differential equations.

③ To formalize the foundations of probability theory, especially in the context of uncountable state spaces (where, typically, any given event has zero probability)

For example, in my own research I am very interested in sampling conditional probability measures on continuous, nonlinear state spaces. Doing so often requires the use of Markov chains &, therefore, a hefty dose of measure theory.

The basic question in measure theory is: given a set (usually a topological space & often $\mathbb{R}^n$, but not always), how do you consistently assign a size (or measure) to subsets?

More precisely, ideally a measure μ on a set X would be a map $\mu: \mathcal{P}(X) \to [0, \infty]$ s.t.

① $\mu(\emptyset) = 0$

② If $E_1, E_2, \ldots \subseteq X$ s.t. $E_i \cap E_j = \emptyset$ for $i \neq j$, then

$$\mu(E_1 \cup E_2 \cup \ldots) = \mu(E_1) + \mu(E_2) + \ldots$$

Unfortunately, this doesn't exactly work unless you do something trivial like declare $\mu(E) = 0 \ \forall \ E \subseteq X$.

Ex: Consider $[0,1]$. I claim there is no way to define a measure in the above sense so that $\mu([0,1]) = c \neq 0$ (obviously, the natural thing would be $c=1$) and so that congruent sets have the same measure.

To see why, for each $x \in [0,1]$, let $E_x := \{y \in [0,1] : y - x \in \mathbb{Q}\}$. The relation $y \sim x \Leftrightarrow y - x \in \mathbb{R}$ is clearly an equivalence relation, so $\forall x, y \in [0,1]$, either $E_x = E_y$ or $E_x \cap E_y = \emptyset$.

Now, use the Axiom of Choice to select one element from each E_x & let V be the set of those numbers.

Also, let $\{r_1, r_2, \dots\}$ be an enumeration of $[0,1] \cap \mathbb{Q}$ & define $V_n := V + r_n = \{x + r_n \pmod 1 : x \in V\}$

Then V & V_n are congruent $\forall n$, so $\mu(V) = \mu(V_n) = d$.

But now $[0,1] = \bigcup_{n=1}^{\infty} V_n$, so $\mu([0,1]) = \mu(\bigcup_{n\geq 1} V_n) = \sum \mu(V_n) = 0$ or ∞, contradicting the desire to have $\mu([0,1]) = c$.

Even worse, the Banach-Tarski paradox says that it is possible to take the unit ball in $\mathbb{R}^3$, chop it into 5 disjoint pieces, and then, using only rigid motions of the pieces, construct two unit balls.

The sets that come up are really nasty (& again require applications of the Axiom of Choice), so the point is that you can't really expect measures to be defined on all subsets of even fairly nice spaces like $[0,1]$ or $\mathbb{R}^3$.

Instead, we will have to specify the "measurable" subsets of our spaces & then only define the measure on those subsets.