Math 617 HW #7

Due 1:00 PM Wednesday, May 4

1. Let $\mathcal{S}(\mathbb{R}^d)$ be the space of Schwartz functions on $\mathbb{R}^d$. Show that the Fourier transform of a Schwartz function is a Schwartz function and that the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ given by $f \mapsto \hat{f}$ is a continuous map.

2. Consider the 2-dimensional torus $T^2 = S^1 \times S^1$, which we can identify with the unit square with opposite sides glued together.

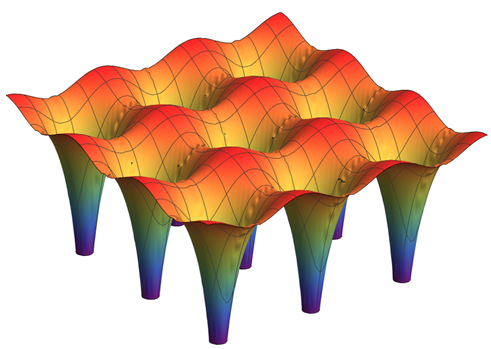
   (a) Under what conditions on $f \in L^2(T^2)$ can the equation

   $$\Delta u = f$$

   be solved? When f satisfies these conditions, write the solution u as explicitly as possible.

   (b) What is the fundamental solution φ of the (scalar) Laplacian on T^2?

   Here's a picture of the graph of (an approximation to) φ over several fundamental domains (so there's really only one singularity, but the graph includes several copies of the unit square):



   (c) Under what conditions on $f \in L^2(T^2)$ can the equation

   $$\nabla \cdot U = f$$

   be solved (U is a vector field)? Find an explicit solution.

   (d) Let V be an L^2 vector field on T^2. Under what conditions can

   $$\nabla f = V$$

   be solved? Find an explicit solution.

   (e) Recall the two-dimensional curl of a vector field $V = (v_1, v_2)$ is the function

   $$\nabla \times U = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.$$

   Under what conditions on $f \in L^2(T^2)$ can

   $$\nabla \times V = f$$

   be solved? Find a solution.