Math 617 HW #6

Due 1:00 PM Friday, April 22

1. Let $c_c(\mathbb{N})$ be the space of sequences of complex numbers with only finitely many nonzero terms (these are "compactly supported" sequences, which is where the subscript c comes from). We can make $c_c(\mathbb{N})$ a normed vector space using the ∞ norm:

$$\|(a_n)_{n=1}^{\infty}\|_{\infty} = \max(a_n).$$

   (a) Show that the dual of $c_c(\mathbb{N})$ is (isomorphic to) $\ell^1(\mathbb{N})$ (which has norm $\|(a_n)\|_{L^1} = \sum |a_n|$).

   (b) Show that the completion of $c_c(\mathbb{N})$ is $c_0(\mathbb{N})$, the space of sequences that go to zero at infinity. This means that $c_0(\mathbb{N})^* \simeq \ell^1(\mathbb{N})$ as well since the dual of a normed vector space is the same as the dual of its closure.

   (c) Show that the dual of $\ell^1(\mathbb{N})$ is (isomorphic to) $\ell^\infty(\mathbb{N})$, which is strictly larger than $c_0(\mathbb{N})$.

   Therefore, since $c_0(\mathbb{N})$ is a Banach space, we see that the double dual of a Banach space need not be isomorphic to the original Banach space.

2. Suppose $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are normed vector spaces and let $T : V \to W$ be a continuous linear transformation with transpose $T^* : W^* \to V^*$.

   (a) If T is surjective, show that T^* is injective.

   (b) If the image of T is dense in W, show that T^* is still injective.

3. (a) Let V be a Banach space. Show that V is reflexive if and only if V^* is reflexive.

   (b) Suppose $(X, \mathcal{M}, \mu)$ is a measure space which contains a countable sequence of disjoint sets of positive measure (for example, $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}(n, n+1)$). Show that $L^1(X, \mathcal{M}, \mu)$ and $L^\infty(X, \mathcal{M}, \mu)$ are not reflexive

4. Let V be a closed subspace of a Hilbert space H. Show that every $x \in H$ has a unique decomposition

$$x = x_V + x_{V^\perp},$$

   where $x_V \in V$ and $x_{V^\perp}$ is orthogonal to every element of V, and

$$\pi_V(x) := x_V$$

   is the closest element of V to x.

5. Prove that all orthonormal bases of a Hilbert space H have the same cardinality.