Math 617 HW #5

Due 1:00 PM Friday, April 8

1. In class we proved the one-sided Hardy–Littlewood maximal inequality in the course of proving the Lebesgue differentiation theorem. Prove the *two-sided* Hardy–Littlewood maximal inequality, which says that, for $f : \mathbb{R} \to \mathbb{C}$ absolutely integrable and $\lambda > 0$,

$$m\left(\left\{x \in \mathbb{R} : \sup_{x \in I} \frac{1}{|I|} \int_I |f(t)| \, \mathrm{d}m(t) \geq \lambda\right\}\right) \leq \frac{2}{\lambda} \int_{\mathbb{R}} |f(t)| \, \mathrm{d}m(t),$$

where I ranges over all intervals of positive length containing x.

2. Given a Lebesgue measurable set $E \subseteq \mathbb{R}^d$, a point $x \in \mathbb{R}^d$ is a *point of density* of E if

$$\frac{m(E \cap B(x,r))}{m(B(x,r))} \to 1 \text{ as } r \to 0.$$

For example, the points of density of $E = [-1,0) \cup (0,1]$ are exactly the points in $(-1,1)$.

Show that, if $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, then almost every point in E is a point of density of E, and almost every point in the complement of E is not a point of density of E.

3. Suppose $(X, \mathcal{M}, \mu)$ is a measure space.

   (a) Let $f \in L^\infty(X, \mathcal{M}, \mu) \cap L^{p_0}(X, \mathcal{M}, \mu)$ for some $0 < p_0 < \infty$. Show that $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$.

   (b) Suppose $f \notin L^\infty(X, \mathcal{M}, \mu)$. Show that $\|f\|_{L^p} \to \infty$ as $p \to \infty$.

4. Let $0 < p, q < \infty$ and let $f \in L^p(X, \mathcal{M}, \mu)$, $g \in L^q(X, \mathcal{M}, \mu)$ so that the inequality in Hölder's inequality is actually an equality. Show that, of the functions f^p and g^q, one is a scalar multiple of the other almost everywhere.

5. Suppose $f \in L^p(X, \mathcal{M}, \mu)$ for some $0 < p \leq \infty$ so that the support of f, $\mathrm{supp}(f) = \{x \in X : f(x) \neq 0\}$, has finite measure. Prove that:

   (a) $f \in L^q(X, \mathcal{M}, \mu)$ for all $0 < q \leq p$;

   (b) $\|f\|_{L^q}^q \to \mu(\mathrm{supp}(f))$ as $q \to 0$.

   (Hence, although the following words don't really mean anything, $\mu(\mathrm{supp}(f))$ is in some sense the 0th power of the L^0 norm of f.)