Math 617 HW #4

Due 1:00 PM Friday, March 25

1. Recall that, when μ is a signed measure, then $\mu = \mu_+ - \mu_-$ and we defined the *absolute value* or *total variation* of μ to be $|\mu| = \mu_+ + \mu_-$. Show that:

   (a) $|\mu|$ is the minimal positive measure such that $-|\mu| \leq \mu \leq |\mu|$;

   (b) $|\mu|(E)$ is the maximum of
   $$\sum_{n=1}^{\infty} |\mu(E_n)|,$$
   where $(E_n)_{n=1}^{\infty}$ ranges over all countable partitions of E into measurable subsets.

2. Let μ be a signed measure on $(X, \mathcal{M})$. Prove that the following three statements are all equivalent:

   (a) μ is a finite signed measure.

   (b) $\mu(E)$ is finite for all $E \in \mathcal{M}$.

   (c) μ_+ and μ_- are both finite positive measures.

3. Let m be Lebesgue measure on $[0, +\infty)$ and let μ be a positive measure. Show that $\mu \ll m$ if and only if the function $x \mapsto \mu([0, x])$ is an absolutely continuous function.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space. A collection $\{f_\alpha : \alpha \in A\} \subseteq L^1(X, \mathcal{M}, \mu)$ is called *uniformly integrable* if, for all $\epsilon > 0$ there exists $\delta > 0$ so that
   $$\left| \int_E f_\alpha d\mu \right| < \epsilon \text{ whenever } \mu(E) < \delta.$$
   Show that:

   (a) Every finite subset of $L^1(X, \mathcal{M}, \mu)$ is uniformly integrable.

   (b) If $(f_n)_{n=1}^{\infty}$ is a sequence of absolutely integrable functions converging (in L^1) to the absolutely integrable function f, then $\{f_n\}$ is uniformly integrable.

5. For $i = 1, 2$, let $(X_i, \mathcal{M}_i)$ be a measurable space and let μ_i, ν_i be σ-finite measures with $\nu_i \ll \mu_i$. Show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and that
   $$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2).$$