# Math 617 HW #3

Due 1:00 PM Friday, March 4

1. Suppose $f \in L^1(\mathbb{R}, \mathcal{L}, m)$ and define $F(x) := \int_{(-\infty,x]} f(t)\,\mathrm{d}m(t)$. Prove that $F$ is continuous.

2. Let $(X, \mathcal{M}_X, \mu)$ be a measure space, let $(Y, \mathcal{M}_Y)$ be a measurable space, and let $\phi : X \to Y$ be a map such that $\phi^{-1}(E) \in \mathcal{M}_X$ for all $E \in \mathcal{M}_Y$ (such maps are sometimes called *measurable morphisms*). Define the *pushforward measure* $\phi_*\mu : \mathcal{M}_Y \to [0, +\infty]$ by

$$\phi_*\mu(E) := \mu(\phi^{-1}(E)).$$

   (a) Show that $\phi_*\mu$ is a measure on $(Y, \mathcal{M}_Y)$.

   (b) If $f : Y \to [0, +\infty]$ is $\phi_*\mu$-measurable, show that

$$\int_Y f\,\mathrm{d}\phi_*\mu = \int_X (f \circ \phi)\,\mathrm{d}\mu.$$

   (c) If $m^n$ is Lebesgue measure on $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation, show that

$$T_* m^n = \frac{1}{|\det T|} m^n.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f_1, f_2, \ldots : X \to [0, +\infty]$ be a sequence of non-negative integrable functions that converge pointwise to an absolutely integrable function $f$. Show that

$$\int_X f_n\,\mathrm{d}\mu - \int_X f\,\mathrm{d}\mu - \|f - f_n\|_{L^1(X,\mathcal{M},\mu)} \to 0$$

as $n \to \infty$. (*Hint:* Consider $\min(f_n, f)$.) (This is telling us that the difference between left and right sides of the inequality in Fatou's Lemma can be estimated by $\|f - f_n\|_{L^1(X,\mathcal{M},\mu)}$.)

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, let $(f_n)$ be a sequence of measurable functions, and let $f$ be measurable. Show that if $f_n$ converges to $f$ in $L^\infty$ norm, then $f_n$ converges to $f$ in $L^1$ norm. (Notice that, as in Egorov's Theorem, the assumption $\mu(X) < \infty$ is essential, as our example $\frac{1}{n}\chi_{[0,n]}$ gives a counterexample on $\mathbb{R}$.)

5. The assumption that $f \in L^1(X \times Y, \mathcal{M}_X \otimes \mathcal{M}_Y, \mu_x \times \mu_y)$ is necessary. To see this, consider $X = \mathbb{N} = Y$ with $\mu_X = \mu_Y$ being counting measure, and let

$$f(x, y) = \begin{cases} 1 & \text{if } x = y \\ -1 & \text{if } x = y + 1 \\ 0 & \text{else.} \end{cases}$$

   Show that the integrals

$$\int_Y f(x, y)\,\mathrm{d}\mu_Y(y) \quad \text{and} \quad \int_X f(x, y)\,\mathrm{d}\mu_X(x)$$

   exist as absolutely integrable functions for all $x \in X$ and $y \in Y$, respectively, and so that

$$\int_X \left( \int_Y f(x, y)\,\mathrm{d}\mu_Y(y) \right) \mathrm{d}\mu_X(x) \quad \text{and} \quad \int_Y \left( \int_X f(x, y)\,\mathrm{d}\mu_X(x) \right) \mathrm{d}\mu_Y(y)$$

   exist as absolutely integrable integrals, but are not equal.

6. Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space and let $f : X \to [0, +\infty]$ be measurable. Recall that $\mathcal{B}_{\mathbb{R}}$ is the Borel $\sigma$-algebra on $\mathbb{R}$ and $m$ is Lebesgue measure on $\mathbb{R}$. Show that

   (a) the set $A = \{(x,t) \in X \times \mathbb{R} : 0 \leq t \leq f(x)\}$ is measurable on $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ and
$$(\mu \times m)(A) = \int_X f(x)\,\mathrm{d}\mu(x);$$

   (b) we have
$$\int_X f(x)\,\mathrm{d}\mu(x) = \int_{[0,+\infty]} \mu(f^{-1}([\lambda, +\infty]))\,\mathrm{d}m(\lambda),$$
   (When $\mu$ is a probability measure, the function inside the right hand side integral is called the *complementary cumulative distribution function.*)

**Bonus Problem** The example in Problem 5 might seem a bit artificial: after all, maybe the discreteness of the setting is what's causing the problem. In fact, similar examples always exist. Give an example of a Borel measurable function $f : [0,1] \times [0,1] \to \mathbb{R}$ so that the integrals
$$\int_{[0,1]} f(x,y)\,\mathrm{d}m(y) \quad \text{and} \quad \int_{[0,1]} f(x,y)\,\mathrm{d}m(x)$$
exist as absolutely integrable functions for all $x \in [0,1]$ and $y \in [0,1]$, respectively, and so that
$$\int_{[0,1]} \left( \int_{[0,1]} f(x,y)\,\mathrm{d}m(y) \right) \mathrm{d}m(x) \quad \text{and} \quad \int_{[0,1]} \left( \int_{[0,1]} f(x,y)\,\mathrm{d}m(x) \right) \mathrm{d}m(y)$$
exist as absolutely integrable integrals, but are not equal.