Math 617 HW #2

Due 1:00 PM Friday, Feb. 19

1. Suppose μ is a Radon measure on $\mathbb{R}$. Show that there exists a monotone nondecreasing function $F : \mathbb{R} \to \mathbb{R}$ so that $\mu = \mu_F$, the Lebesgue–Stieltjes measure associated with F.

2. Let $F, G : \mathbb{R} \to \mathbb{R}$ be monotone nondecreasing functions. Show that the associated Lebesgue–Stieltjes measures μ_F and μ_G are the same if and only if there exists a constant C so that $F_+(x) = G_+(x) + C$ and $F_-(x) = G_-(x) + C$ for all $x \in \mathbb{R}$. (Note that this means the value of F at points of discontinuity is irrelevant to μ_F, and indeed $\mu_F = \mu_{F_+} = \mu_{F_-}$.)

3. We know that Lebesgue measure m on $\mathbb{R}$ is a Radon measure, which implies that for any Lebesgue measurable set $E \subseteq \mathbb{R}$,

$$m(E) = \sup_{K \subseteq E, K \text{ compact}} m(K).$$

   Give an example to show that it is *not* necessarily the case that

$$m(E) = \sup_{U \subseteq E, U \text{ open}} m(U).$$

4. Let $f : \mathbb{R} \to [0, +\infty]$ be Lebesgue measurable and assume $\varphi : [0, +\infty] \to [0, +\infty]$ is continuous. Prove that $\varphi \circ f : \mathbb{R} \to [0, +\infty]$ is Lebesgue measurable.

5. Suppose $\{f_n\}$ is a sequence of measurable functions on a measurable space $(X, \mathcal{M})$. Show that the set

$$A := \{x : \lim_{n \to \infty} f_n(x) \text{ exists}\}$$

   is measurable.

6. Suppose $(X, \mathcal{M}, \mu)$ is a measure space. In general, we say that some property holds μ-*almost everywhere* if the set of $x \in X$ on which it fails to hold has measure zero.

   Show that the following are equivalent:

   (a) μ is complete

   (b) If f is measurable and $f = g$ μ-almost everywhere (i.e., the set $\{x \in X : f(x) \neq g(x)\}$ has measure zero), then g is measurable.

   (c) If f_n is measurable for all $n \in \mathbb{N}$ and $f_n \to f$ μ-almost everywhere, then f is measurable.