Math 617 HW #1

Due 1:00 PM Friday, Feb. 5

1. Let X be a set, let I be a (possibly uncountable) index set and suppose that, for all $\alpha \in I$, $\mathcal{A}_\alpha$ is a σ-algebra on X. Show that
$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha$$
is a σ-algebra on X. (Recall that this was the key to defining the σ-algebra generated by a given collection of subsets of X.)

2. Suppose $(X, \mathcal{M})$ is a measurable space and assume $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$. If $a_1, \ldots, a_n$ are nonnegative real numbers, show that
$$\sum_{i=1}^{n} a_i \mu_i$$
is also a measure on $(X, \mathcal{M})$.

3. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and that $E, F \in \mathcal{M}$. Show that
$$\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F).$$

4. Recall that m_* and m^* are the Jordan inner content and Jordan outer content, respectively. Show that, if $E \subset \mathbb{R}$ and $\overline{E}$ is the closure of E and E° is the interior of E, then
   (a) $m^*(E) = m^*(\overline{E})$
   (b) $m_*(E) = m_*(E^\circ)$

5. Define a *Jordan null set* to be a Jordan measurable set whose Jordan content is zero. Show that any subset of a Jordan null set is itself a Jordan null set (and in particular is Jordan measurable).

6. Recall that we can extend the notion of elementary sets to $\mathbb{R}^d$ for any d as follows. We define a *box* to be any set B such that $B = I_1 \times \ldots \times I_d$ where the I_j are intervals, and we define the volume of the box to be
$$|B| = |I_1| \cdot |I_2| \cdot \cdots \cdot |I_d|.$$
Then the elementary sets are disjoint unions of boxes and the elementary measure $m(E)$ of an elementary set $E = B_1 \cup \ldots \cup B_n$ is simply $m(E) := |B_1| + \ldots + |B_n|$. In turn, we define inner and outer Jordan content as in the 1-dimensional case:
$$m_*(E) := \inf_{A \subset E, A \text{ elementary}} m(A)$$
$$m^*(E) := \sup_{B \supset E, B \text{ elementary}} m(B)$$
and a set $E \subset \mathbb{R}^d$ is *Jordan measurable* if $m(E) := m_*(E) = m^*(E)$.

(a) Show that the closed ball $B(x, r) := y \in \mathbb{R}^d : |y - x| \leq r$ is Jordan measurable, with Jordan content

$$m(B(x, r)) = c_d r^d$$

for some constant $c_d > 0$ depending only on d.

(b) Prove the crude bounds

$$\left(\frac{2}{\sqrt{d}}\right)^d \leq c_d \leq 2^d.$$

(In fact, it turns out that $c_d = \frac{2\pi^{d/2}}{d\Gamma(d/2)}$ where Γ is the gamma function.)