Math 517 Midterm Exam Practice Problem Solution Sketches

1. For each of the following statements, say whether it is true or false. If the statement is true, explain why; if it is false, give a counterexample. Your counterexamples should be as explicit as possible, but if you can't give an explicit counterexample, at least explain how one might go about constructing an explicit counterexample.

   (a) Every continuous function is differentiable.
   Answer: False. Consider $f(x) = |x|$.

   (b) If X is a metric space, every Cauchy sequence in X converges.
   Answer: False. Let $X = \mathbb{Q}$ and $\{a_n\}$ a sequence of rational numbers converging to $\sqrt{2}$ (e.g., successive decimal approximations or the recursively-defined series $a_1 = 2$, $a_{n+1} = \frac{a_n^2+2}{a_n}$).

   (c) If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.
   Answer: True. Since continuous images of compact sets are compact, $f([a, b])$ must be closed and bounded.

   (d) If $\{a_n\}$ is a sequence of real numbers so that $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.
   Answer: False. Let $a_n = \frac{1}{n}$.

   (e) If $\sum a_n$ converges, then $\sum |a_n|$ converges.
   Answer: False. Let $a_n = \frac{(-1)^n}{n}$.

   (f) If X and Y are metric spaces, $f : X \to Y$, and $E \subseteq X$ is closed, then $f(E)$ is closed in Y.
   Answer: False. Let $X = Y = \mathbb{R}$, $E = [0, 1]$, and $f(x) = \begin{cases} x & x \le 1/2 \\ 2 - 2x & x > 1/2 \end{cases}$.

   (g) If $\{z_n\}$ is a sequence of complex numbers so that $\{|z_n|\}$ converges, then $\{z_n\}$ converges.
   Answer: False. Let $z_n = \frac{(-1)^n}{n}$.

   (h) A function is continuous if and only if it satisfies the intermediate value property.
   Answer: False. Consider $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

2. Give examples of each of the following, or explain why no such example exists.

   (a) An infinite compact set.
   Answer: $[0, 1] \subseteq \mathbb{R}$.

   (b) A continuous function which is not differentiable on its whole domain.
   Answer: $f(x) = |x|$.

   (c) A finite open set in $\mathbb{R}^2$.
   Answer: The empty set.

   (d) An unbounded closed subset of $\mathbb{R}$.
   Answer: $[0, +\infty)$.

   (e) A set of real numbers with exactly three limit points.
   Answer: $(0, 1) \cup (1, 2)$.

3. Using only the definition of continuity, show that $f(x) = x^2$ is continuous on all of $\mathbb{R}$.

   Proof. Let $x \in \mathbb{R}$ and let $\epsilon > 0$. Then for $|x - y| < \min(1, \frac{\epsilon}{2|x|+1})$, we have

   $$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq (2|x| + 1)|x - y| < (2|x| + 1)\frac{\epsilon}{2|x| + 1} = \epsilon,$$

   so f is continuous at x. $\square$

4. Determine the radius of convergence of the power series

   $$\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n}} x^n.$$

   Answer: Using the ratio test,

   $$\alpha = \limsup \left| \frac{\frac{2^{n+1}}{\sqrt{n+1}}}{\frac{2^n}{\sqrt{n}}} \right| = \limsup 2\sqrt{\frac{n}{n+1}} = 2,$$

   so the radius of convergence is $R = \frac{1}{\alpha} = \frac{1}{2}$.

5. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable and that $f'(x) > 0$ for all $x \in (a, b)$. Show that f is strictly increasing.

   Proof. Suppose $a \leq x < y \leq b$. By the Mean Value Theorem, there exists $x < c < y$ so that

   $$f(y) - f(x) = f'(c)(y - x) > 0,$$

   so we see that $f(y) > f(x)$. Since the choice of x and y was arbitrary, we see f is strictly increasing on $[a, b]$. $\square$

6. Let X be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Define $E = \{x \in X : f(x) = 0\}$. Show that E is compact.

   Proof. Since $\{0\} \subseteq \mathbb{R}$ is closed, E is the inverse image of a closed set under a continuous map, and hence must be closed in X. But all closed subsets of a compact set are compact, so E is compact. $\square$