Math 517 Exam #1 Practice Problems

1. For each of the following statements, say whether it is true or false. If the statement is true, explain why; if it is false, give a counterexample. Your counterexamples should be as explicit as possible, but if you can't give an explicit counterexample, at least explain how one might go about constructing an explicit counterexample.

   (a) Every continuous function is differentiable.

   (b) If X is a metric space, every Cauchy sequence in X converges.

   (c) If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.

   (d) If $\{a_n\}$ is a sequence of real numbers so that $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.

   (e) If $\sum a_n$ converges, then $\sum |a_n|$ converges.

   (f) If X and Y are metric spaces, $f : X \to Y$, and $E \subseteq X$ is closed, then $f(E)$ is closed in Y.

   (g) If $\{z_n\}$ is a sequence of complex numbers so that $\{|z_n|\}$ converges, then $\{z_n\}$ converges.

   (h) A function is continuous if and only if it satisfies the intermediate value property.

2. Give examples of each of the following, or explain why no such example exists.

   (a) An infinite compact set.

   (b) A continuous function which is not differentiable on its whole domain.

   (c) A finite open set in $\mathbb{R}^2$.

   (d) An unbounded closed subset of $\mathbb{R}$.

   (e) A set of real numbers with exactly three limit points.

3. Using only the definition of continuity, show that $f(x) = x^2$ is continuous on all of $\mathbb{R}$.

4. Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n}} x^n.$$

5. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable and that $f'(x) > 0$ for all $x \in (a, b)$. Show that f is strictly increasing.

6. Let X be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Define $E = \{x \in X : f(x) = 0\}$. Show that E is compact.