Math 474: Day 40

We continue w/ the proof of the second part of

Clairaut's Relation: The geodesics on a surface of revolution satisfy $r\cos\varphi = \text{const.}$ where r is the distance to the axis & φ is the angle w/ a parallel. Moreover, any constant speed curve satisfying $r\cos\varphi = \text{const}$ which is not a parallel is a geodesic.

Proof: we want to show that $|\alpha'(t)| = \text{const.}$ & $r\cos\varphi = \text{const} \Rightarrow \alpha(t)$ is either a parallel or a geodesic.

If $\text{const} = |\alpha'(t)| = \langle u'\vec{x}_u + v'\vec{x}_v, u'\vec{x}_u + v'\vec{x}_v\rangle = (u')^2 \underbrace{E}_{=2} + 2u'v'\underbrace{F}_{\to 0} + (v')^2 \underbrace{G}_{=f(u)^2} = (u')^2 + (v')^2 f(u)^2$, then we can diff'te w.r.t. t to get

$$0 = 2u'u'' + 2f(u)f'(u)u'(t)(v')^2 + 2f(u)^2 v'v''$$

If $r\cos\varphi = \text{const}$, then we can run the previous argument backward to deduce

$$v'' = -\frac{2f'u'v'}{f} \leftarrow 2^{\text{nd}} \text{ geodesic eqn!}$$

which we can substitute into the above to get

$$0 = 2u'u'' + 2f(u)f'(u)u'(t)(v')^2 - 4f(u)f'(u)u'(v')^2$$

$$= 2u'\left(u'' + ff'(v')^2 - 2ff'(v')^2\right)$$

$$= 2u'\left(\underbrace{u'' - ff'(v')^2}_{1^{\text{st}} \text{ geodesic eqn!}}\right)$$

So either $u' \equiv 0$ (meaning the curve is a parallel), or α satisfies both geodesic eqns, & hence is a geodesic. ∎

Holonomy & Gauss-Bonnet

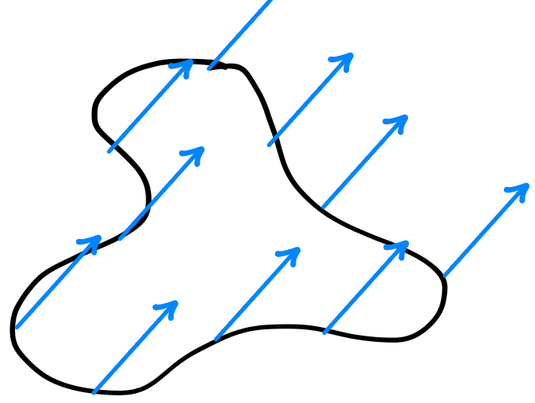
The idea here is that: ① Curvature is a measure of how parallel transport can alter a vector

② Curvature reflects the topology of a surface

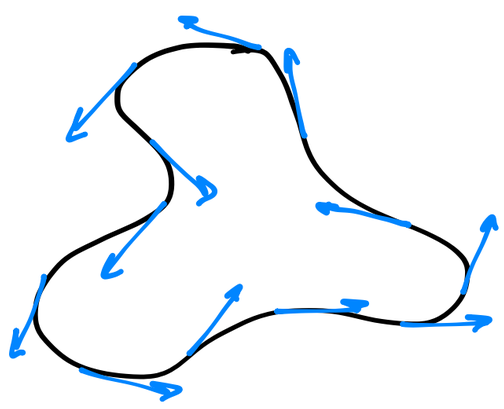
Def: The **holonomy** of a unit vector field V on a closed curve $\alpha \subset \Sigma$ is given by $\int_\alpha \langle V', \vec{n}\times V\rangle\, ds$

This gives the total twisting of the frame $\{V, \vec{n}\times V\}$ for the tangent plane $T_{\alpha(s)}\Sigma$ as we traverse the loop.

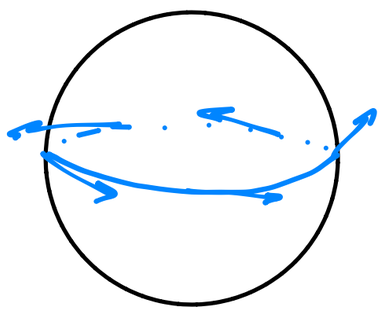
Ex: ① A unit vector field on a plane curve has zero holonomy

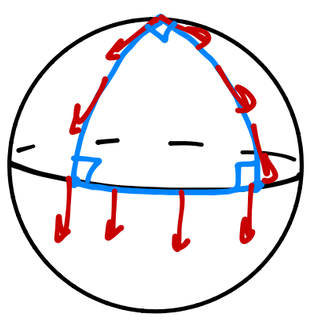


② The tangent vector on a simple closed plane curve has holonomy 2π



③ The tangent vector to a great circle on the sphere has holonomy 0



In this case $V' = \alpha''$ is large, set orthgl to $\vec{n} \times V = \vec{n} \times \alpha'$.

④ The parallel transport of a vector around the spherical right triangle bounding the positive octant has holonomy $\pi/2$