Math 474 HW #7

Due 2:00 PM Friday, Nov. 10

1. (Shifrin Problem 2.2.14) Suppose that for every $p \in \Sigma$, the shape operator S_p is some scalar multiple of the identity; i.e., $S_p(\vec{v}) = k(p)\vec{v}$ for all $\vec{v} \in T_p\Sigma$. (The scalar $k(p)$ is allowed to depend on the point p.)

   (a) Differentiate the equations

   $$D_{\vec{x}_u}\vec{n} = \vec{n}_u = -k\vec{x}_u$$
   $$D_{\vec{x}_v}\vec{n} = \vec{n}_v = -k\vec{x}_v$$

   appropriately to determine k_u and k_v and deduce that k must be constant.

   (b) Show that when $k \neq 0$, Σ is (a portion of) a sphere.

   (Notice that this shows that a surface with all points umbilic and positive Gauss curvature must be a part of a sphere, which was a key component of our proof of Liebmann's theorem.)

2. (Shifrin Problem 2.3.2) Calculate the Christoffel symbols for the following parametrized surfaces. Then check in each case that the Codazzi equations and the first Gauss equation hold.

   (a) The plane, parametrized by polar coordinates $\vec{x}(u, v) = (u \cos v, u \sin v, 0)$.

   (b) A helicoid, $\vec{x}(u, v) = (u \cos v, u \sin v, v)$.

   (c) A cone, $\vec{x}(u, v) = (u \cos v, u \sin v, cu)$, $c \neq 0$.

   (d) A surface of revolution, $\vec{x}(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$, with $f'(u)^2 + g'(u)^2 = 1$ for all u.

3. (Shifrin Problem 2.3.12) Show that the Gaussian curvature of the parametrized surfaces

   $$\vec{x}(u, v) = (u \cos v, u \sin v, v)$$
   $$\vec{y}(u, v) = (u \cos v, u \sin v, \ln u)$$

   is the same for each (u, v), but that the first fundamental forms $I_{\vec{x}}$ and $I_{\vec{y}}$ do not agree.