Math 474 HW #5

Due 2:00 PM Friday, Oct. 20

1. Find the first fundamental form (i.e., E, F, and G) of the following surfaces:
   (a) The sphere of radius r: $\vec{x}(u,v) = r(\sin u\cos v, \sin u \sin v, \cos u)$
   (b) The torus: $\vec{x}(u,v) = ((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u)$ $(0 < b < a)$.
   (c) The helicoid: $\vec{x}(u,v) = (u\cos v, u\sin v, bv)$.
   (d) The catenoid: $\vec{x}(u,v) = (a\cosh u\cos v, b\cosh u \sin v, u)$.

2. Consider the sphere $x^2+y^2+z^2 = 1$ centered at the origin in $\mathbb{R}^3$. We can construct a very important map st from the unit sphere to the xy-plane by defining $\mathrm{st}(p)$ to be the intersection of the line through $p \in S^2$ and $(0,0,1)$ with the xy-plane (of course this map is not well defined at the point $(0,0,1)$).
   (a) Show that the inverse map $\mathrm{st}^{-1} : \mathbb{R}^2 \to S^2\backslash\{(0,0,1)\}$ is given by
   $$\mathrm{st}^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$
   (b) Show that st^{-1} is a regular parametrization of $S^2\backslash\{(0,0,1)\}$.
   (c) Use st^{-1} and the corresponding map $\mathbb{R}^2 \to S^2\backslash\{(0,0,-1)\}$ to prove (again) that S^2 is a regular surface.

3. (Shifrin Problem 2.1.6) A parametrization $\vec{x}(u,v)$ of a surface Σ is called *conformal* if angles measured in the uv-plane agree with the corresponding angles in $T_p\Sigma$ for all points $p \in \Sigma$. Prove that the parametrization $\vec{x}(u,v)$ is conformal if and only if $E = G$ and $F = 0$.

4. (Shifrin Problem 2.1.8) Show that the parametrization of the unit sphere by inverse stereographic projection (see Problem 2) is conformal.

   Shown below is what you get when you map a square grid in the plane to the sphere by inverse stereographic projection. Notice that all the lines intersect perpendicularly.