Math 474: Week 8

Differentials

For a regular surface Σ w/ parametrization $\vec{x}$, we let $\vec{n} = \dfrac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}$ be the unit normal vector to Σ. The tangent plane to Σ at a point $\vec{p} \in \Sigma$ is denoted $T_{\vec{p}}\Sigma$.

Now, if we have a differentiable map $f : \Sigma_1 \to \Sigma_2$, then we get a corresponding linear map

$$df_{\vec{p}} : T_{\vec{p}}\Sigma_1 \to T_{f(\vec{p})}\Sigma_2$$

Abstractly, this is a linear map from the plane to the plane, so we should be able to write it as a 2×2 matrix.

Of course, to do so we need bases for $T_{\vec{p}}\Sigma_1$ & for $T_{f(\vec{p})}\Sigma_2$.

But it's easy to get bases from our parametrizations (a.k.a., local coordinates). If a nbhd of $\vec{p} \in \Sigma_1$ is parametrized by $\vec{x}_1 : U_1 \subseteq \mathbb{R}^2 \to \Sigma_1$ w/ (u_1, v_1) coordinates on the domain, then

$$T_{\vec{p}}\Sigma_1 = \operatorname{span}\{\vec{x}_{u_1}, \vec{x}_{v_1}\} \text{ shifted to pass thru } \vec{p}.$$

So $\vec{x}_{u_1}$ & $\vec{x}_{v_1}$ give natural coords. on $T_{\vec{p}}\Sigma_1$:

$$\vec{w}_1 \in T_{\vec{p}}\Sigma_1 = w_{11}\vec{x}_{u_1} + w_{12}\vec{x}_{v_1} + \vec{p}$$

for some numbers w_1 & w_2.

In practice, since we want to think of $T_{\vec{p}}\Sigma_1$ as a linear space, we will forget about shifting by $\vec{p}$ & just think of $\vec{x}_{u_1}$ & $\vec{x}_{v_1}$ as determining a basis for $T_{\vec{p}}\Sigma_1$.

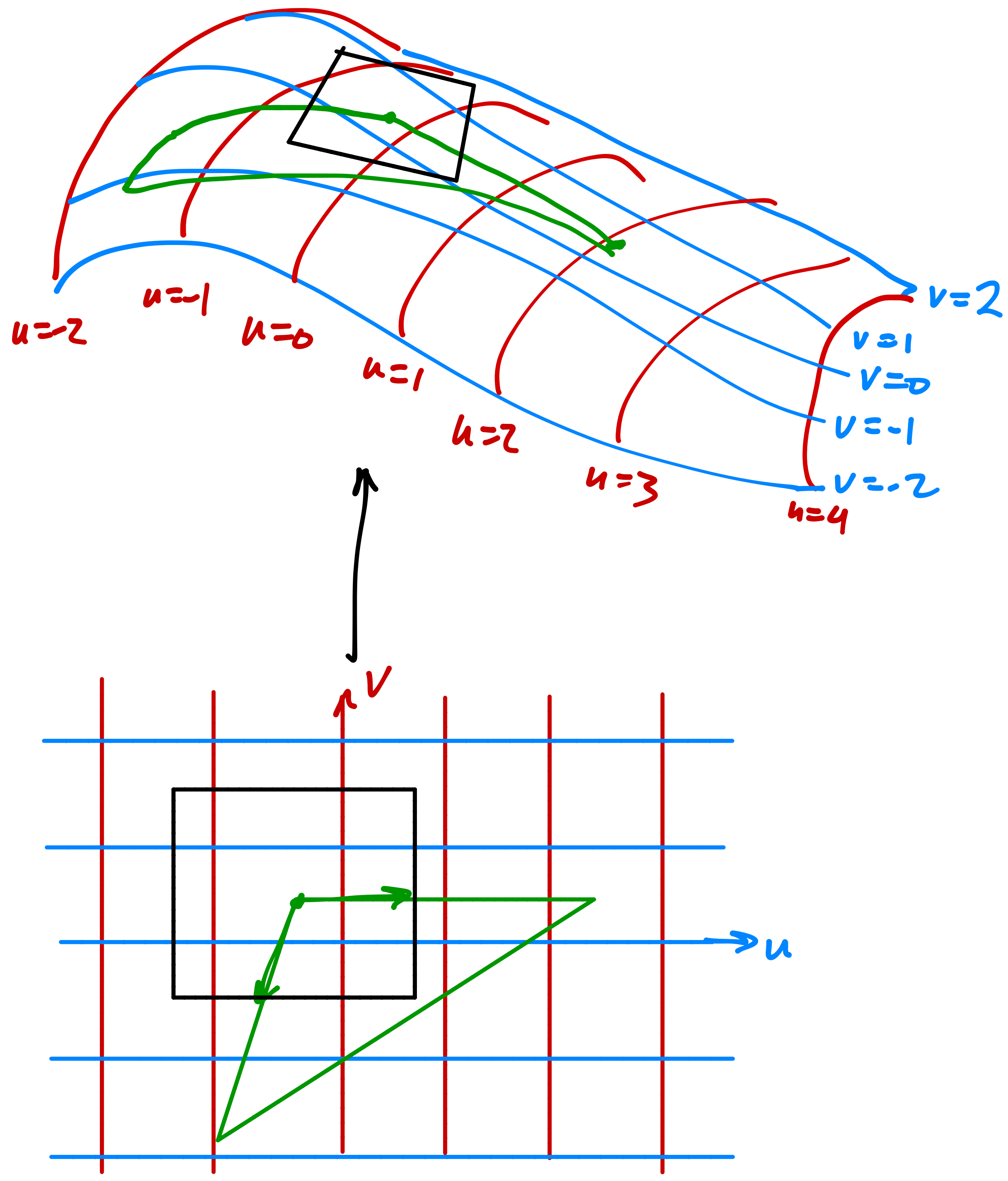
So now, if we have local coords. (u_1, v_1) on Σ_1 & (u_2, v_2) on Σ_2 (induced by a parametrization $\vec{x}_2 : U_2 \subseteq \mathbb{R}^2 \to \Sigma_2$) then we can write f in local coords. as $f(u_1, v_1) = (u_2(u_1, v_1), v_2(u_1, v_1))$ & then, w.r.t. the (w_{11}, w_{12}) & (w_{21}, w_{22}) coords. on $T_{\vec{p}}\Sigma_1$ & $T_{f(\vec{p})}\Sigma_2$, respectively, we have

$$df_{\vec{p}} = \begin{bmatrix} \frac{\partial u_2}{\partial u_1} & \frac{\partial u_2}{\partial v_1} \\ \frac{\partial v_2}{\partial u_1} & \frac{\partial v_2}{\partial u_2} \end{bmatrix}.$$

(Of course the specific entries we get in this matrix depend on the choice of local coordinates, but it can be shown that the underlying linear map is independent of these choices)

Geometry of Regular Surfaces

We've now defined regular surfaces, tangent planes, & local coordinates.



The parametrization gives our local coords. u & v on the surface Σ. Now, we know how to do geometry in the plane, & the goal is to use that knowledge to allow us to measure length, angles, and areas with respect to the curved image Σ of the plane.

This will require some algebraic ideas that may not be so familiar, so we pause for a ...

Brief Review of (or Introduction to) Quadratic Forms

First, we'll recall some linear algebra. For this we'll use the notation $\langle \vec{v}, \vec{w} \rangle$ for the inner product of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ (a.k.a. the dot product: $\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w}$).

Then it is not very hard to see that, for any $n\times n$ matrix A, we have $\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A^T\vec{w} \rangle$ (really, this is why we care about transposes)

Now, if A is a symmetric matrix, meaning $A = A^T$, then we can define a quadratic form associated to A:

$$Q_A(\vec{v}, \vec{w}) = \langle \vec{v}, A\vec{w} \rangle$$

Now, this form is a function on pairs of vectors which is

① symmetric: $Q_A(\vec{v}, \vec{w}) = Q_A(\vec{w}, \vec{v})$ (b/c $A = A^T$)

② bilinear: $Q_A(a\vec{v}_1 + b\vec{v}_2, w) = a\,Q_A(\vec{v}_1, \vec{w}) + b\,Q_A(\vec{v}_2, \vec{w})$

Furthermore, if all the matrices have positive determinant, then Q_A is

③ positive-definite: $Q_A(\vec{v},\vec{v}) \geq 0$ w/ equality $\iff \vec{v} = \vec{0}$.

Here is one use of a quadratic form: Given vectors
$$\vec{v} = v_1\vec{b}_1 + \dots + v_n\vec{b}_n$$
$$\vec{w} = w_1\vec{b}_1 + \dots + w_n\vec{b}_n$$

written in terms of an arbitrary basis $\vec{b}_1, \dots, \vec{b}_n$ for a subspace $\mathbb{R}^n$ of $\mathbb{R}^k$, how can we compute $\vec{v} \cdot \vec{w}$?

By linearity,
$$\vec{v} \cdot \vec{w} = \sum_{i,j=1}^{n} v_i w_j (\vec{b}_i \cdot \vec{b}_j)$$

or, if A is the Gram matrix $\begin{bmatrix} \vec{b}_1\cdot\vec{b}_1 & \cdots & \vec{b}_n\cdot\vec{b}_1 \\ \vdots & \ddots & \vdots \\ \vec{b}_1\cdot\vec{b}_n & \cdots & \vec{b}_n\cdot\vec{b}_n \end{bmatrix}$, then

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i \sum_{j=1}^{n} A_{ij} w_j = \langle \vec{v}, A\vec{w} \rangle .$$

This is really the geometric interpretation of the matrix A:

A_{ij} = inner product of $\vec{e}_i$ & $\vec{e}_j$ according to the Q_A inner product.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then $Q_A(\vec{v},\vec{w}) = (1,2) \cdot A(1,1) = (1,2) \cdot (3,7) = 17$

So now, w/ quadratic forms in our toolbag, back to the geometry of surfaces.

If $\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a parametrized surface, then at any point $p_0 = (u_0, v_0)$ in the uv-plane we have the quadratic form determined by

$$A_{p_0} = \begin{bmatrix} \vec{x}_u\cdot\vec{x}_u & \vec{x}_u\cdot\vec{x}_v \\ \vec{x}_v\cdot\vec{x}_u & \vec{x}_v\cdot\vec{x}_v \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \longleftarrow \text{this is usually how people write this matrix.}$$

Def: The **first fundamental form** (of the parametrization $\vec{x}$) at $p=(u_0,v_0)$ is the quadratic form

$I_p(\vec{v},\vec{w})$ (or sometimes $\langle \vec{v},\vec{w}\rangle_p$ to emphasize that it's an inner product on the tangent space at p)

determined by the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$. (Note: the I is supposed to evoke a Roman numeral 1 because this is the first fundamental form)

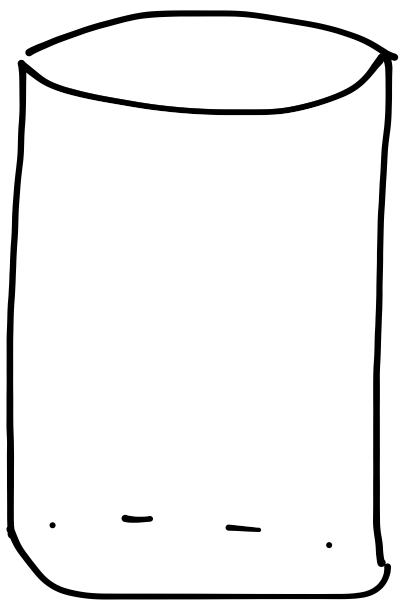
Of course, since $\vec{x}_u$ & $\vec{x}_v$ (and hence E,F,G) depend on p, this matrix changes from point to point.

Prop: I_p is positive-definite.

Proof: We know that $I_p(\vec{w},\vec{w}) = (w_1\vec{x}_u + w_2\vec{x}_v)\cdot(w_1\vec{x}_u + w_2\vec{x}_v)$. But this is just a standard dot

product of vectors in $\mathbb{R}^3$, so $I_p(\vec{w},\vec{w}) \geq 0$ with $I_p(\vec{w},\vec{w}) = 0 \Leftrightarrow \vec{w}=\vec{0}$. ▨

Ex: Consider the cylinder



which has parametrization $\vec{x}(u,v) = (\cos u, \sin u, v)$

Now, $\vec{x}_u = \begin{bmatrix} -\sin u \\ \cos u \\ 0 \end{bmatrix}$ & $\vec{x}_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so we can compute

$E = \vec{x}_u\cdot\vec{x}_u = \sin^2 u + \cos^2 u = 1$, $F = \vec{x}_u\cdot\vec{x}_v = 0$ & $G = \vec{x}_v\cdot\vec{x}_v = 1$, so the matrix is just

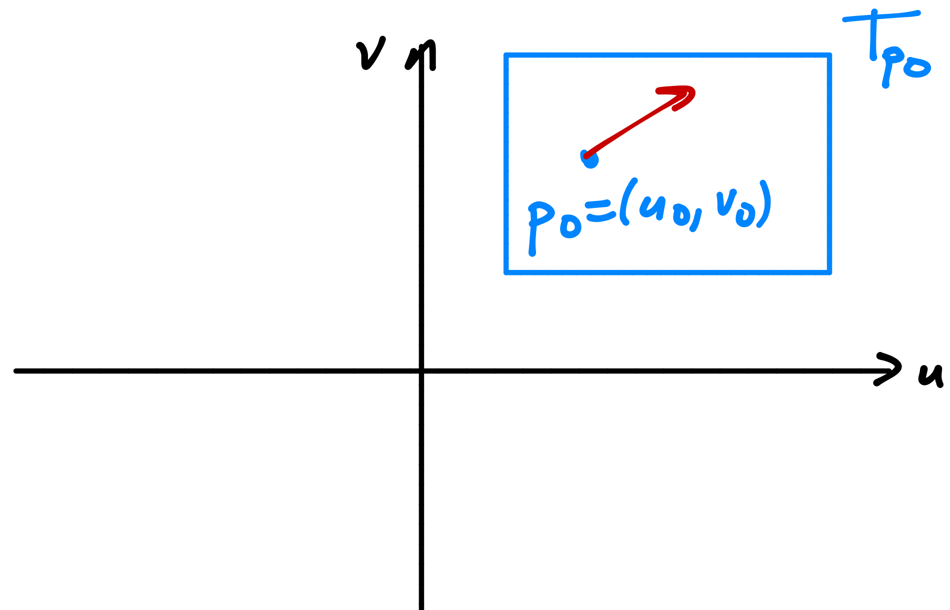
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & the first fundamental form is just the usual dot product on the uv-plane.

Ex: The xy-plane (sitting inside $\mathbb{R}^3$) is parametrized by $\vec{x}(u,v) = (u,v,0)$.

Then $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\vec{x}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so $E=1$, $F=0$, $G=1$, the same as the cylinder (this turns out not to be a coincidence)

Computing with I_p

We now need to make an important (but subtle) distinction:



$T_{p_0}\mathbb{R}^2$, the tangent plane to $\mathbb{R}^2$ at p_0, where $\vec{x}(p_0) = \vec{p} \in \mathbb{R}^3$

← Parameter space $\mathbb{R}^2$ maps into $\mathbb{R}^3$ by $\vec{x}$.

The tangent plane $T_{p_0}\mathbb{R}^2$ to the uv-plane is a vector space on which we put the inner product I_p.

We can obviously measure the length of vectors $\vec{v} \in T_{p_0}\mathbb{R}^2$ with I_p: $\|\vec{v}\|_p = \sqrt{I_p(\vec{v},\vec{v})}$

Now the issue is that the uv-plane itself does not have a useful inner product... only the spaces $T_{p_0}\mathbb{R}^2$ do.

So we need to measure length by integration. If $\alpha(t): [0,\ell] \to \mathbb{R}^2$ is given by $\alpha(t) = (u(t), v(t))$ is a parametrized curve in $\mathbb{R}^2$ (and hence, by composing w/ $\vec{x}$, on our surface), then

$$\text{Length}(\alpha) = \int_0^\ell \sqrt{I_p(\alpha'(t),\alpha'(t))}\, dt = \int_0^\ell \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt,$$

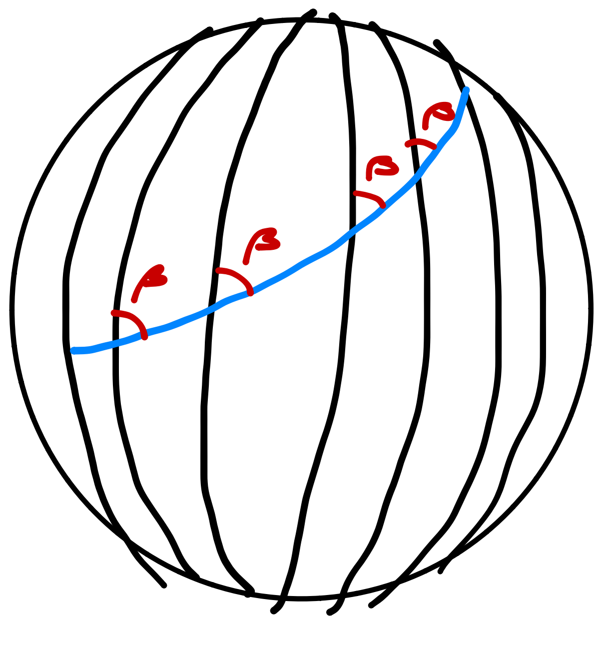
Sometimes written "$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$".

We can also define the angle θ between $\vec{w}_1, \vec{w}_2 \in T_{p_0}\mathbb{R}^2$ by

$$\cos\theta = \frac{I_p(\vec{w}_1,\vec{w}_2)}{\sqrt{I_p(\vec{w}_1,\vec{w}_1)\, I_p(\vec{w}_2,\vec{w}_2)}}$$

Consequently, the parametrization basis vectors $\vec{x}_u, \vec{x}_v \in T_p\Sigma$ are orthogonal $\iff F = I_p((1,0),(0,1)) = 0$.

Ex: Let $\vec{x}(\varphi,\theta) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ parametrize the sphere. Find the equation of a curve in the (θ,φ) plane (& hence on the sphere) which makes a constant angle with the curves $\varphi = \text{constant}$ (the lines of longitude):



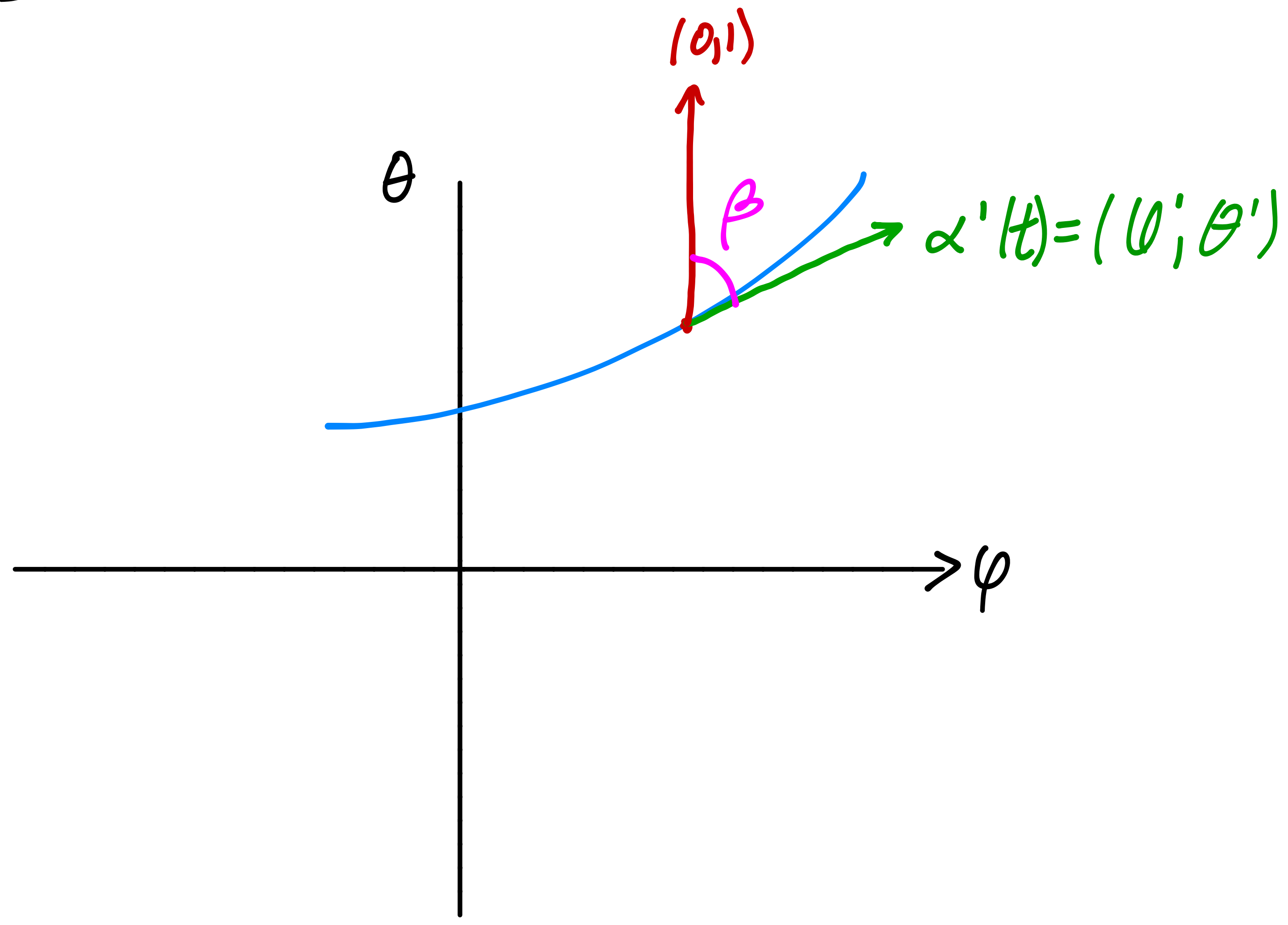
We need to compute $E = \vec{x}_\varphi \cdot \vec{x}_\varphi$, $F = \vec{x}_\varphi \cdot \vec{x}_\theta$, & $G = \langle \vec{x}_\theta, \vec{x}_\theta \rangle$:

$$E = \vec{x}_\varphi \cdot \vec{x}_\varphi = \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix} = \sin^2\theta$$

$$G = \vec{x}_\theta \cdot \vec{x}_\theta = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \underline{1}$$

$$F = \vec{x}_\varphi \cdot \vec{x}_\theta = \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = 0$$

In the $\varphi\theta$-plane the picture is:



Now, $$\cos\beta = \frac{I_p((\varphi',\theta'),(0,1))}{\|\alpha'(t)\|_p \, \|(0,1)\|_p}$$

$$= \frac{(\varphi',\theta')\cdot\begin{bmatrix} E & F \\ F & G\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}}{1\sqrt{(\varphi',\theta')\cdot\begin{bmatrix} E & F \\ F & G\end{bmatrix}\begin{bmatrix}\varphi'\\ \theta'\end{bmatrix}}}$$

$$= \frac{\theta'}{\sqrt{\sin^2\theta(\varphi')^2+(\theta')^2}}$$

So we have that $$(\cos^2\beta)(\sin^2\theta(\varphi')^2+(\theta')^2) = (\theta')^2$$

$$(\cos^2\beta)(\varphi')^2 = \frac{(\theta')^2(1-\cos^2\beta)}{\sin^2\theta}$$

or $$\frac{\varphi'}{\tan\beta} = \pm\frac{\theta'}{\sin\theta}$$

Integrating both sides with respect to t gives

$$\frac{\varphi}{\tan\beta} + C = \pm\ln(\tan(\theta/2)) \quad \text{or} \quad \varphi = \pm(\tan\beta)(\ln(\tan(\theta/2))) + C$$

where C is determined by the starting point.

(The integration comes from the half-angle formula $\sin\theta = \sin(2\tfrac{\theta}{2}) = 2\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})$, so

$$\int\frac{\theta'(t)\,dt}{2\sin\theta/2\,\cos\theta/2} \underset{u=\theta/2}{=} \int\frac{1}{\sin u\cos u}\,du = \int\frac{\cos u}{\sin u}\cdot\frac{1}{\cos^2 u}\,du = \int\frac{\sec^2 u}{\tan u}\,du = \ln(\tan u) = \ln(\tan(\theta/2))$$)

Finally, consider area on surfaces. In $\mathbb{R}^3$, the area spanned by $\vec{v}$ & $\vec{w}$ is $|\vec{v}\times\vec{w}|$. Using this yields:

Def: If $R \subseteq U$ is a bounded region in the parameter plane of a regular surface given by $\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, then

$$\mathrm{Area}(\vec{x}(R)) := \mathrm{Area}(R) = \iint_R |\vec{x}_u \times \vec{x}_v| \, du\, dv$$

Now, we use the very handy identity

$$|\vec{x}_u \times \vec{x}_v|^2 + \langle \vec{x}_u, \vec{x}_v \rangle_p^2 = \|\vec{x}_u\|_p^2 \, \|\vec{x}_v\|^2$$

to write

$$\mathrm{Area}(R) = \iint_R \sqrt{EG - F^2} \, du\, dv$$

where $\sqrt{EG-F^2}$ gets called the *element of area*. Notice that this quantity is the square root of the determinant of the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ which maps $T_{p_0}\mathbb{R}^2 \to T_{\vec{p}}\Sigma$.