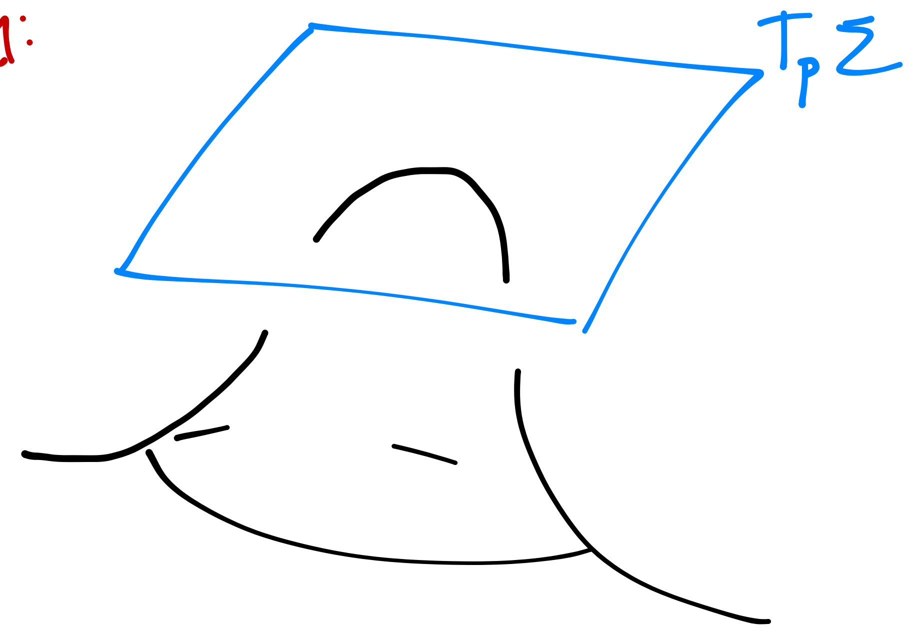
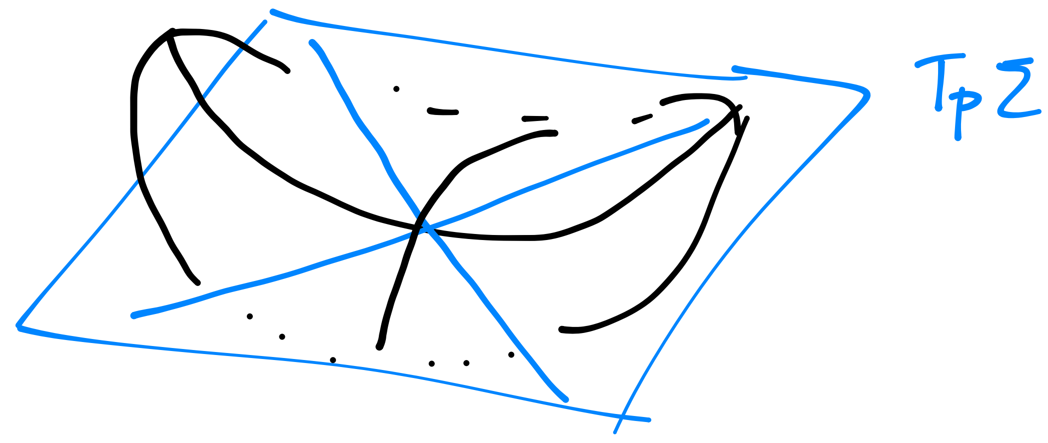
Math 474: Week 11

Let's see how to extract geometric information from the first & second fundamental forms.

Claim:





$T_p\Sigma$ lies on one side of Σ

$\Updownarrow$

elliptic

$T_p\Sigma$ cuts Σ

$\Updownarrow$

hyperbolic

To build up, let's think of II_p in a slightly different way.

When we defined I_p, we proved that the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ & the quadratic form I_p were positive-definite, meaning that

$$I_p(\vec{v},\vec{v}) \geq 0 \quad \& \quad I_p(\vec{v},\vec{v}) = 0 \iff \vec{v} = \vec{0}.$$

Recall: A matrix M is positive definite $\iff$ all eigenvalues of M are >0.

A form Q is negative definite if $Q(\vec{v},\vec{v}) \leq 0$ & $Q(\vec{v},\vec{v}) = 0 \iff \vec{v} = \vec{0}$.

In general, a form is definite if it is either positive or negative definite, & it is indefinite if $\exists\ \vec{v},\vec{w}$ s.t.

$$Q(\vec{v},\vec{v}) < 0 \quad \& \quad Q(\vec{w},\vec{w}) > 0.$$

So now

p is elliptic

$\Updownarrow$

$K(p) > 0 \iff k_1k_2 > 0$

$\Updownarrow$

II_p is definite

p is hyperbolic

$\Updownarrow$

$K(p) < 0 \iff k_1k_2 < 0$

$\Updownarrow$

II_p is indefinite

At parabolic points, II_p is semidefinite, meaning either $II_p(\vec{v},\vec{v}) \geq 0\ \forall \vec{v}$ or $II_p(\vec{v},\vec{v}) \leq 0\ \forall \vec{v}$.

So now we can formalize the claim as

Prop: · If $p \in \Sigma$ is an elliptic point, $\exists$ nbhd $\mathcal{U}$ of p s.t. all points in $\mathcal{U}$ lie on the same side of $T_p\Sigma$.

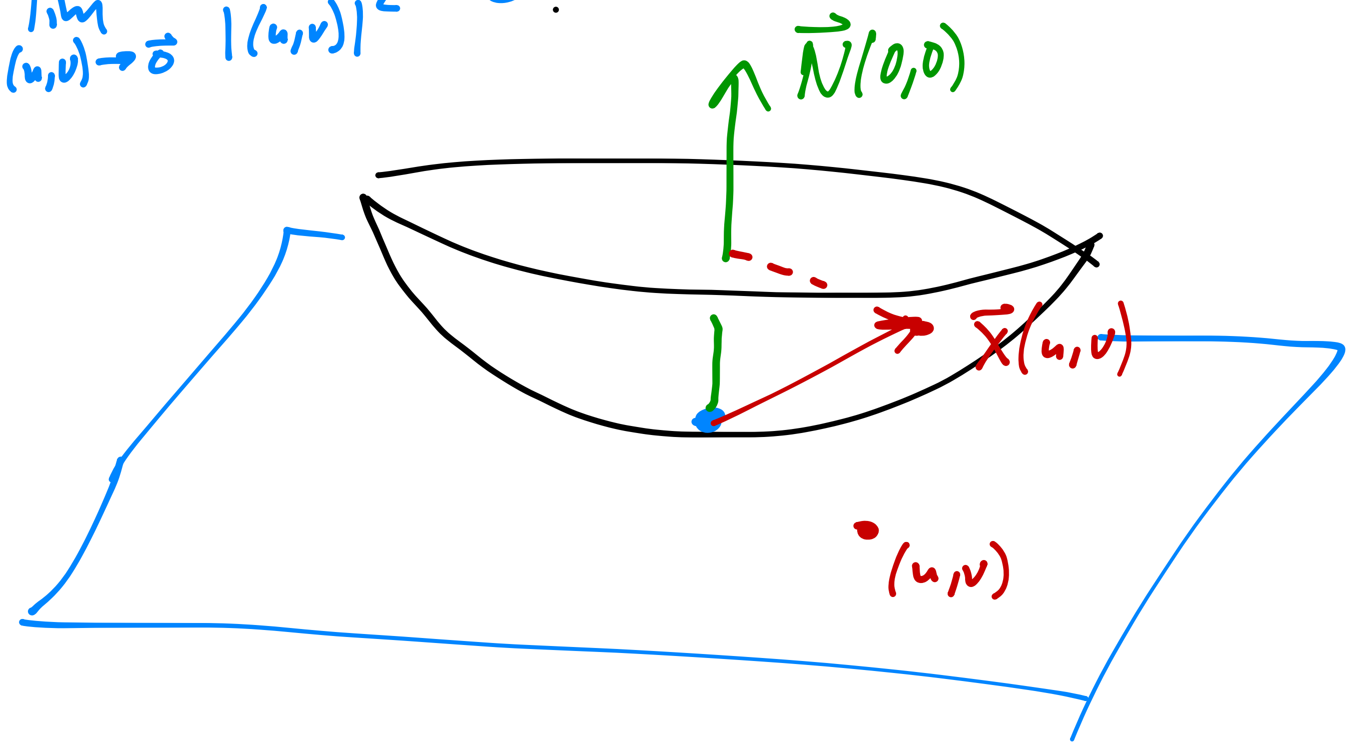
· If p is a hyperbolic point, then any nbhd of p contains pts. on both sides of $T_p\Sigma$.

Proof: Suppose $p = \vec{x}(0,0) = (0,0,0)$ (which we can always achieve by translation & rotation).

Then, by Taylor's Thm

$$\vec{x}(u,v) = \vec{x}_u u + \vec{x}_v v + \tfrac{1}{2}\left(\vec{x}_{uu} u^2 + 2\vec{x}_{uv} uv + \vec{x}_{vv} v^2\right) + \vec{R}(u,v),$$

where $\lim\limits_{(u,v)\to\vec{0}} \dfrac{\vec{R}(u,v)}{|(u,v)|^2} = \vec{0}$.



The height of $\vec{x}(u,v)$ above the tangent plane is given by

$$d = \langle \vec{x}, \vec{N}\rangle = \langle \vec{x}_u, \vec{N}\rangle u + \langle \vec{x}_v, \vec{N}\rangle v + \tfrac{1}{2}\left(\langle \vec{x}_{uu}, \vec{N}\rangle u^2 + 2\langle \vec{x}_{uv}, \vec{N}\rangle uv + \langle \vec{x}_{vv}, \vec{N}\rangle v^2\right) + \langle \vec{R}, \vec{N}\rangle$$

$$= 0 + 0 + \tfrac{1}{2}\left(eu^2 + 2fuv + gv^2\right) + \langle \vec{R}, \vec{N}\rangle$$

$$= \tfrac{1}{2}\begin{bmatrix} u & v\end{bmatrix}\begin{bmatrix} e & f \\ f & g\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix} + \langle \vec{R}, \vec{N}\rangle$$

$$= \tfrac{1}{2}\,\mathrm{II}_p\left(\begin{bmatrix} u \\ v\end{bmatrix}\right) + \langle \vec{R}, \vec{N}\rangle \qquad \text{(So cool!)}$$

If p is an elliptic point, then II_p is definite, so $\mathrm{II}_p(\vec{w},\vec{w})$ has the same sign for all $\vec{w} \in T_p\Sigma$.

OTOH, if p is hyperbolic, then II_p is indefinite & $\mathrm{II}_p(\vec{w},\vec{w})$ changes sign.

For small enough (u,v), we can ignore the $\vec{R}$ terms & this proves the result.

Go back to: asymptotic directions $\iff \mathrm{II}_p(\vec{v},\vec{v}) = 0$. Then the curve $\alpha(t) = (u(t), v(t))$ is an asymptotic curve iff

$$(u')^2 e + (u'v')\cdot 2f + (v')^2 g \equiv 0,$$

which is a particular differential eqn for $\alpha(t)$.

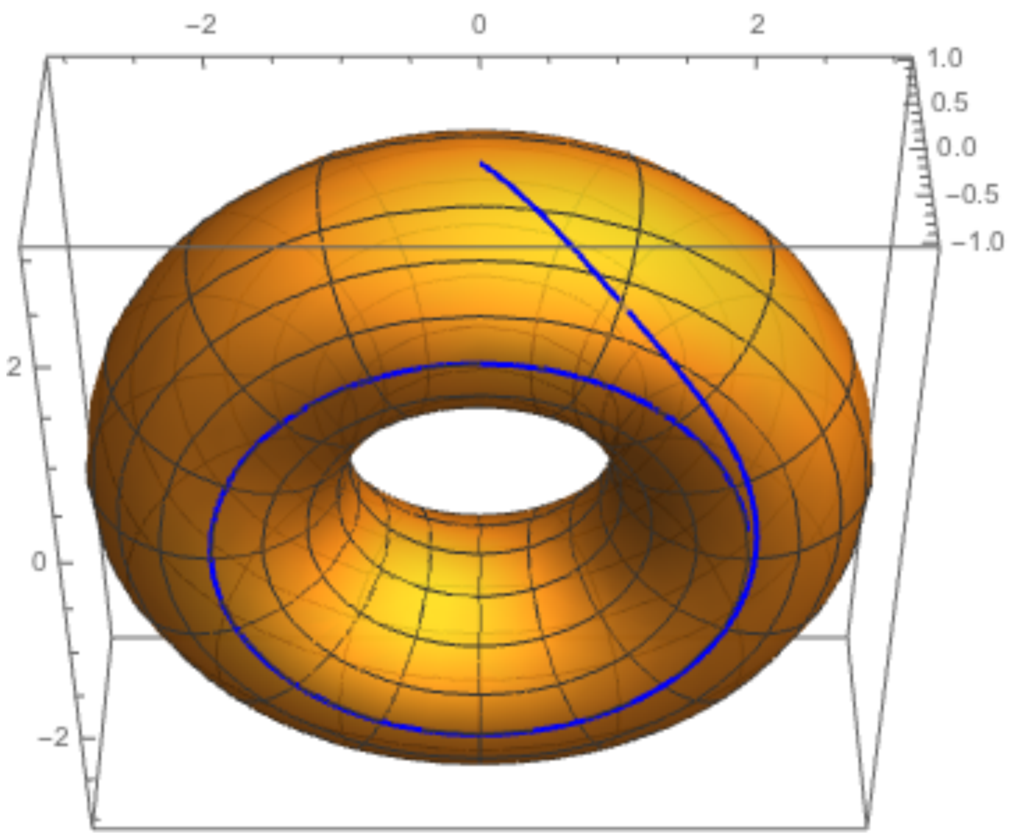
Ex: On the torus of revolution, we saw that

$$e=r, \quad f=0, \quad g=\cos u(a+r\cos u)$$

So $\alpha(t)$ is an asymptotic curve $\Leftrightarrow$ $(u')^2 r + (v')^2 \cos u(a+r\cos u) = 0$ or, equivalently,

$$\frac{d}{dv}(u(v)) = \frac{u'}{v'} = -\frac{\cos u}{r}(a+r\cos u)$$

which we could solve for $u(v)$:



What about principal directions?

Recall: $\alpha(t)$ is a line of curvature $\Leftrightarrow$ $d\vec{N}_p(\alpha'(t)) = \lambda(t)\alpha'(t)$ or, if $\alpha'(t) = (u'(t), v'(t))$, then

$$a_{11}u' + a_{12}v' = \lambda u' \qquad (1)$$
$$a_{21}u' + a_{22}v' = \lambda v' \qquad (2)$$

We can solve by multiplying (1) by v', multiplying (2) by u', and subtracting to get

$$-a_{21}(u')^2 + (a_{12}-a_{22})u'v' + a_{12}(v')^2 = 0$$

or, using the Weingarten equations

$$(fE-eF)(u')^2 + (gE-eG)u'v' + (gF-fG)(v')^2 = 0$$

Ex: On the torus of revolution,

$$E=r^2 \qquad F=0 \qquad G=(a+r\cos u)^2$$

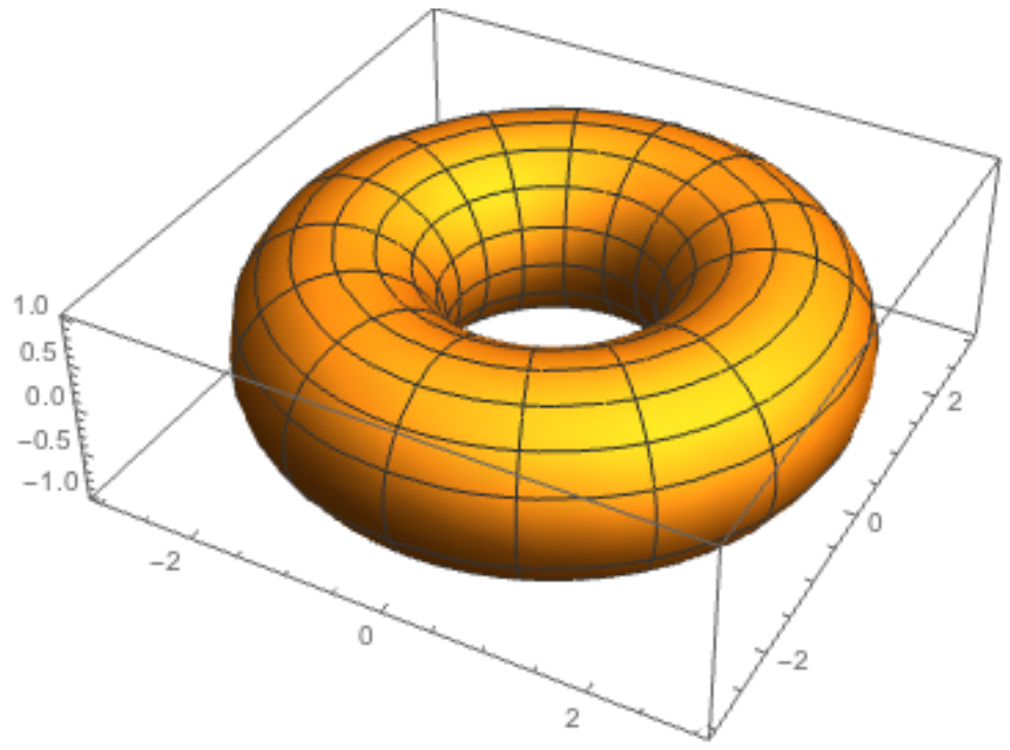
So we get

$$\left(r^2\cos u(a+r\cos u) - r(a+r\cos u)^2\right)u'v' = 0 \Leftrightarrow r(a+r\cos u)\left(r\cos u - (a+r\cos u)\right)u'v' = 0$$

or

$$-r(a+r\cos u)\, a\, u'v' = 0$$

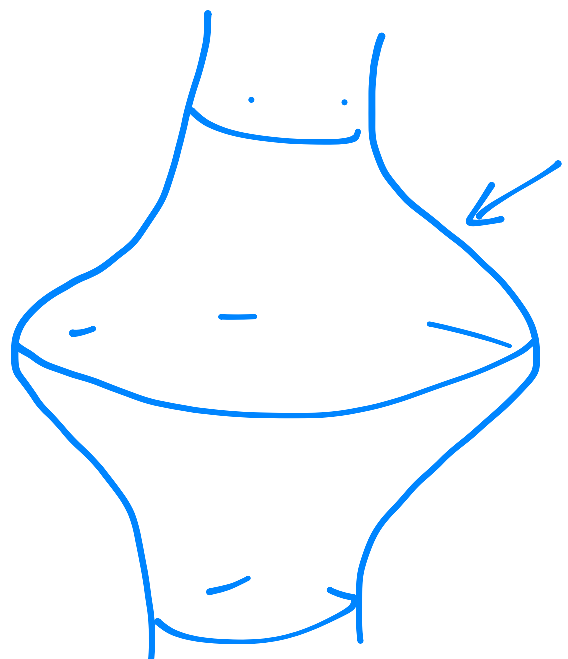
But then this means $u'=0$ or $v'=0$ and the lines of curvature are just the coordinate lines



Note: The key part of the above calculation was that $f=F=0$. In fact:

coordinate lines are lines of curvature $\iff f=F=0$.

Ex: Surfaces of revolution



$\alpha(v)=(\varphi(v),\psi(v))$

$$\vec{X}(u,v)=(\varphi(v)\cos u,\ \varphi(v)\sin u,\ \psi(v))$$

To get the coeffs E,F,G,e,f,g, we compute

$$\left.\begin{aligned}\vec{X}_u&=(-\varphi(v)\sin u,\ \varphi(v)\cos u,\ 0)\\ \vec{X}_v&=(\varphi'(v)\cos u,\ \varphi'(v)\sin u,\ \psi'(v))\end{aligned}\right\}\Rightarrow E=\varphi(v)^2 \quad F=0 \quad G=(\varphi')^2+(\psi')^2=|\alpha'(v)|^2$$

$$\vec{X}_{uu}=(-\varphi(v)\cos u,\ -\varphi(v)\sin u,\ 0)$$

$$\vec{X}_{uv}=(-\varphi'(v)\sin u,\ \varphi'(v)\cos u,\ 0)$$

$$\vec{X}_{vv}=(\varphi''(v)\cos u,\ \varphi''(v)\sin u,\ \psi''(v))$$

If we assume α is parametrized by arclength, then $G=1$, so

$$\sqrt{EG-F^2}=\varphi$$

Also,

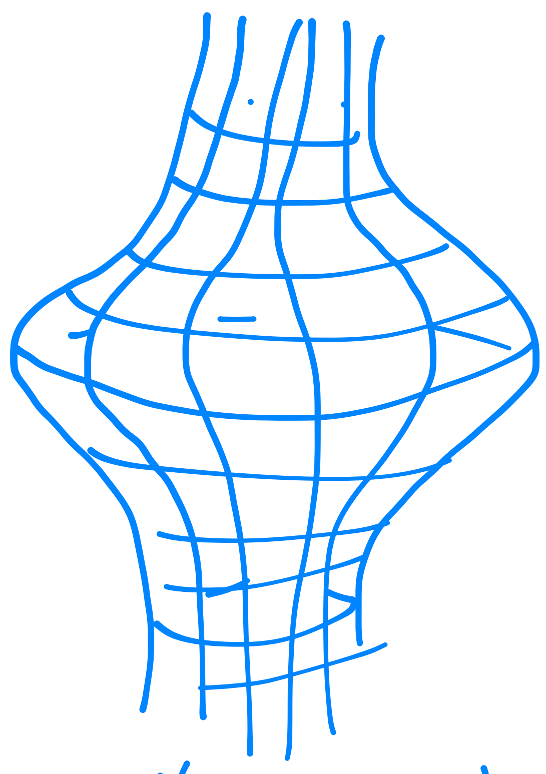
$$e=\frac{1}{\sqrt{EG-F^2}}\langle\vec{X}_u\times\vec{X}_v,\vec{X}_{uu}\rangle=\frac{1}{\varphi}\begin{vmatrix}-\varphi\sin u & \varphi\cos u & 0\\ \varphi'\cos u & \varphi'\sin u & \psi'\\ -\varphi\cos u & -\varphi\sin u & 0\end{vmatrix}=\frac{1}{\varphi}\left(-\psi'\left(\varphi^2\sin^2 u+\varphi^2\cos^2 u\right)\right)=-\varphi\psi'$$

Similarly, $f=0$ & $g=\psi'\varphi''-\psi''\varphi'$.

But of course g is familiar. We know that the curvature of $\alpha(t)$ is given by

$$\kappa(t) = \frac{|\alpha'(t) \times \alpha''(t)|^2}{|\alpha'(t)|^3} = \frac{|\varphi'\psi'' - \psi'\varphi''|}{1} = |\varphi'\psi'' - \psi'\varphi''|$$

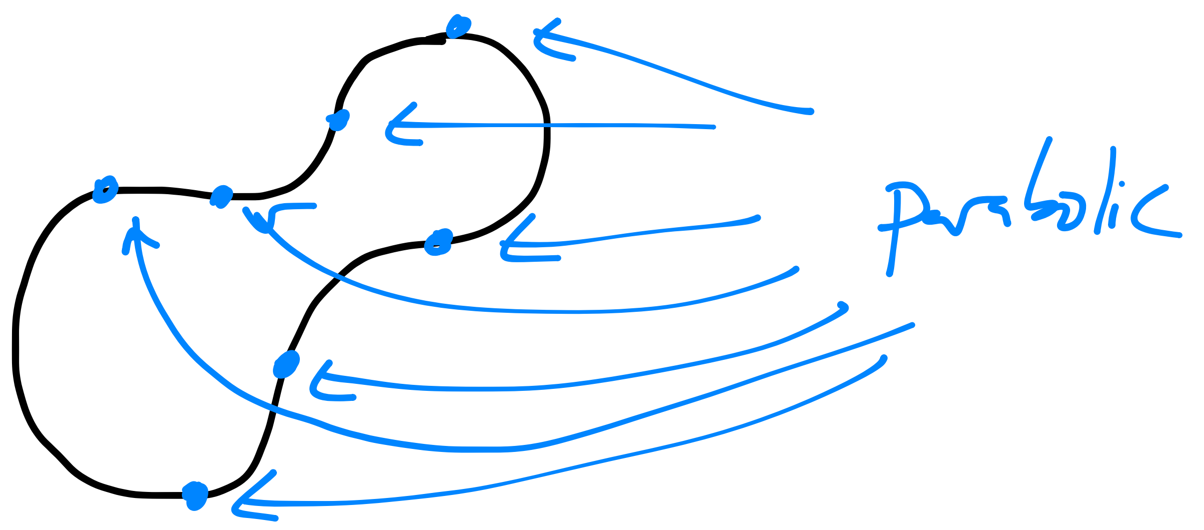
Since $f = F = 0$, we know the coordinate curves are again lines of curvature



Also: $K = \frac{eg - f^2}{EG - F^2} = \frac{-\varphi\psi'(\psi'\varphi'' - \psi''\varphi')}{\varphi^2} = -\frac{1}{\varphi}\psi'(\psi'\varphi'' - \psi''\varphi')$

So where are the parabolic points?

$K = 0 \iff \psi' = 0$ or $\psi'\varphi'' - \psi''\varphi' = 0$ (the curve has a point of zero curvature)



Of course, there's a better expression for K:

differentiating

$$\varphi'^2 + \psi'^2 = 1$$

w.r.t. v yields

$$2\varphi'\varphi'' + 2\psi'\psi'' = 0 \quad \text{or} \quad \varphi'\varphi'' = -\psi'\psi'',$$

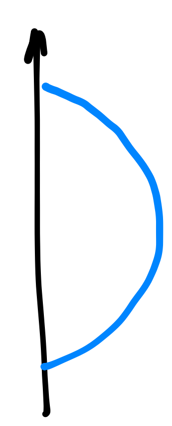
So $$K = \frac{-(\psi')^2\varphi'' + \psi'\psi''\varphi'}{\varphi} = \frac{(-(\psi')^2 - (\varphi')^2)\varphi''}{\varphi} = -\frac{\varphi''}{\varphi}$$

Application: A surface of revolution has constant Gaussian curvature $\iff -\frac{\varphi''}{\varphi} = K_0$ or $\varphi'' = -K_0\varphi$

Solutions of this ODE are simple:

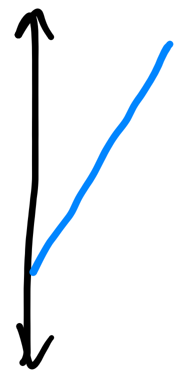
$K_0 = +1 \iff \varphi'' = -\varphi \iff \varphi(v) = \sin v$ or $\cos v$

$\psi(v) = \cos v$ or $\sin v$



A sphere!

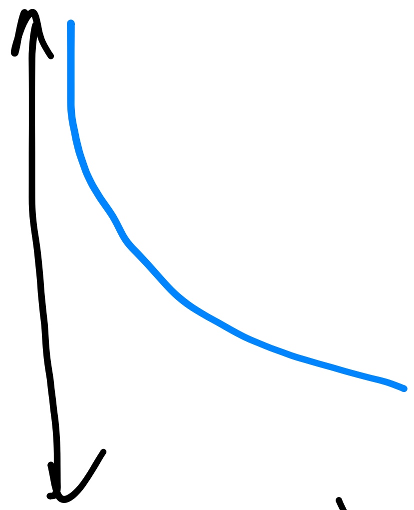
$K_0 = 0 \iff \varphi'' = 0 \iff \varphi(v) = av + b$



A cone or cylinder

$K_0 = -1 \iff \varphi'' = \varphi \iff \varphi(v) = \cosh v$ or $\sinh v$

$\psi(v) = \int \sqrt{1 - \sinh^2 v}\, dv$



Pseudosphere!

(Q: Is there a nicer parametrization for the pseudosphere than the ~~above~~ φ & ψ?)