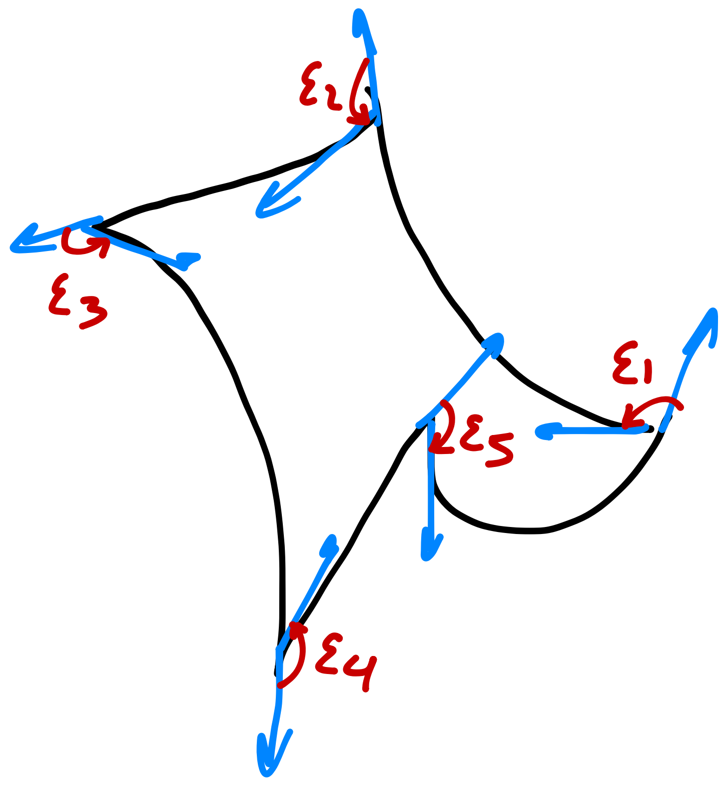
Math 474: Days 39-40

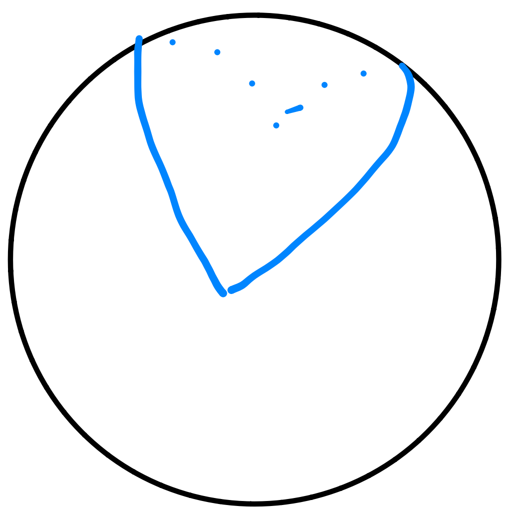


(Local) Gauss-Bonnet theorem:

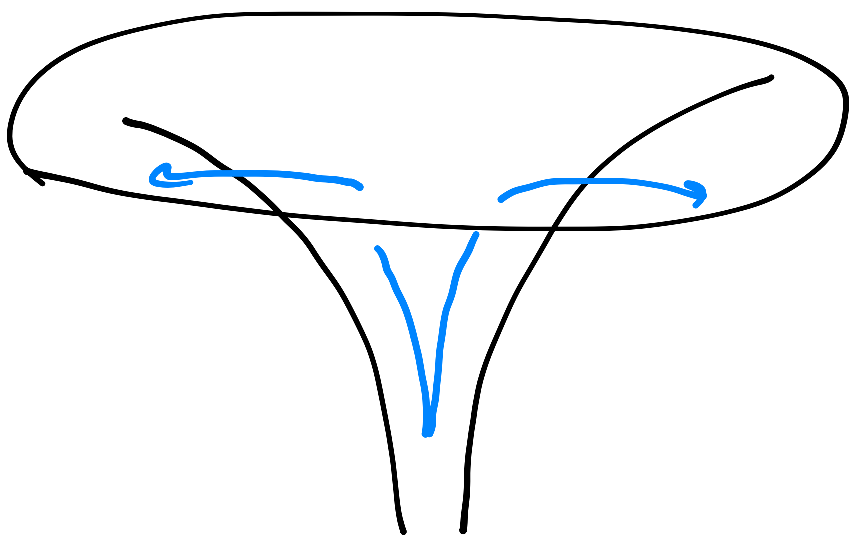
$$\sum \int_{\alpha_i} \kappa_g(s)\,ds + \iint_R K\, d\text{Area} = 2\pi - \sum \varepsilon_i .$$

We'll use this theorem to prove the global Gauss-Bonnet Thm, but note that it also tells us something about geodesics.

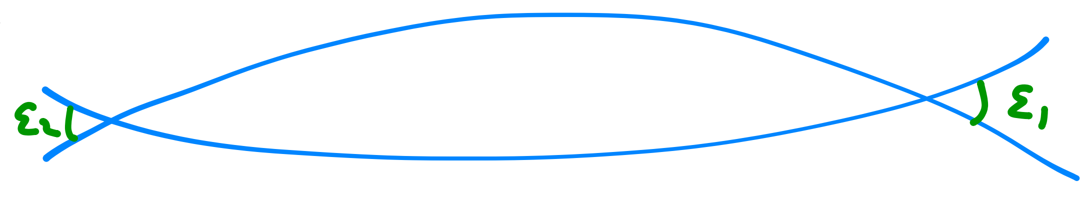
We know on a surface of positive curvature (like the sphere) that geodesics come back together



But on surfaces of negative curvature this can't happen (unless there's some topology). Specifically, on a surface of negative curvature, 2 geodesics **cannot** intersect twice & bound a simply-connected region



Proof: If 2 geodesics met twice & bounded a simply connected region, like so:



But then, by Gauss-Bonnet,

$$\iint_R K = 2\pi - \varepsilon_1 - \varepsilon_2$$

Since $\varepsilon_1, \varepsilon_2 < \pi$, the RHS is >0. But $K<0$, so $\iint K < 0$ $(\Rightarrow\Leftarrow)$ ∎

Now, to prove the global Gauss-Bonnet thm, which says:

Global Gauss-Bonnet Thm

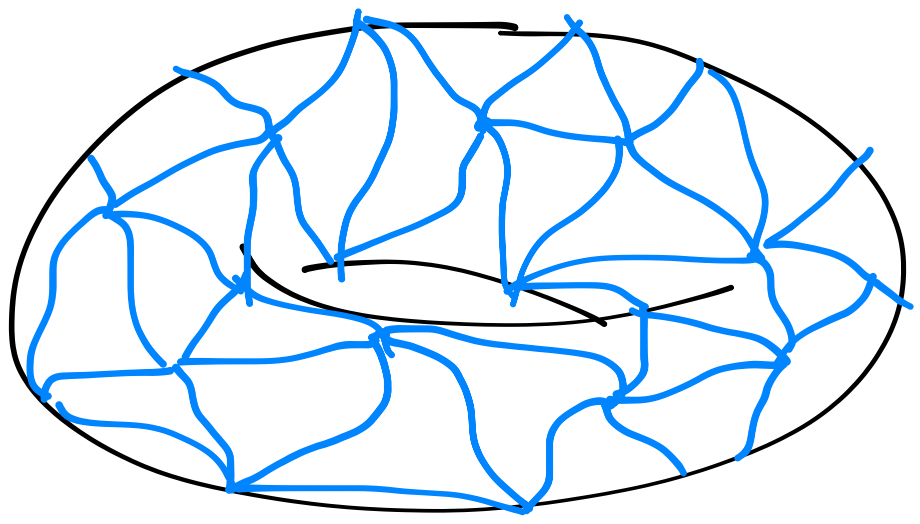
If Σ is a compact, orientable surface w/o boundary, then

$$\iint_\Sigma K \, dArea = 2\pi \chi_T(\Sigma)$$

for any triangulation T of Σ.

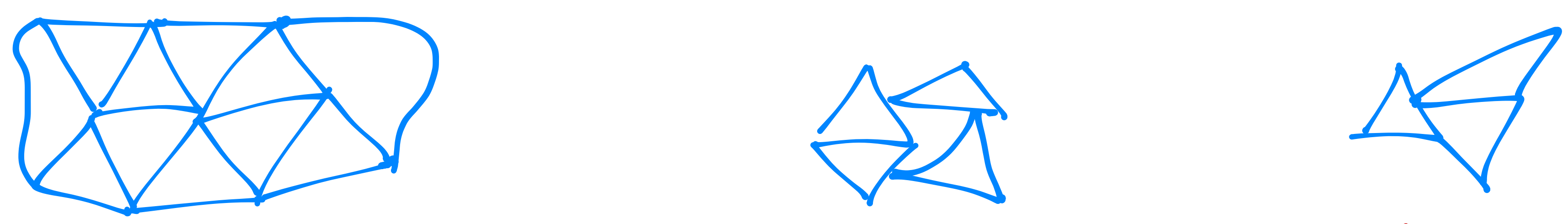
Of course, we have to say what χ_T is.

The idea is to tile Σ by triangles.

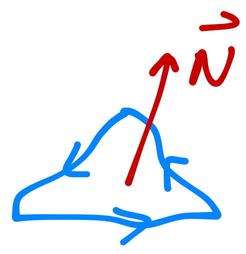


Def: A surface Σ has a triangulation $T = \{\Delta_i \subseteq \Sigma\}$ if each Δ_i is the image of a triangle in the plane under a local coordinate map & if $\Delta_i \cap \Delta_j$ is always either empty, a common vertex, or a common edge

Ex:



Orient the boundary of each triangle so that the interior is to the left



Now, given a triangulation we define

F = # of triangles

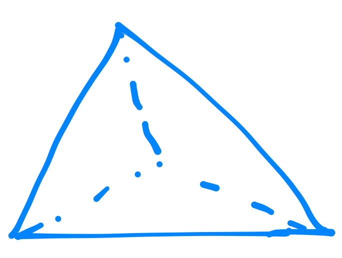
E = # of edges

V = # of vertices

& define the Euler characteristic $\chi_T(\Sigma)$ of the triangulated surface Σ to be

$$\chi_T(\Sigma) = V - E + F$$

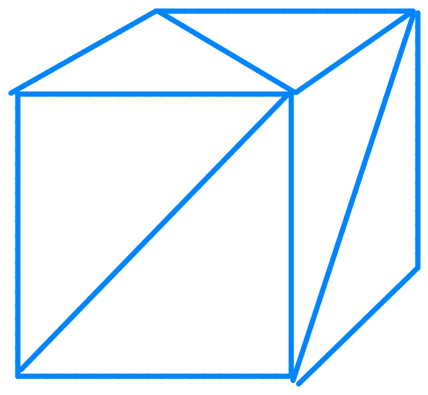
Ex: Tetrahedron



$V=4 \quad E=6 \quad F=4$

so $\chi = 2$

Ex: Cube



$V=8$

$E=18$ $\qquad \chi=2$

$F=12$

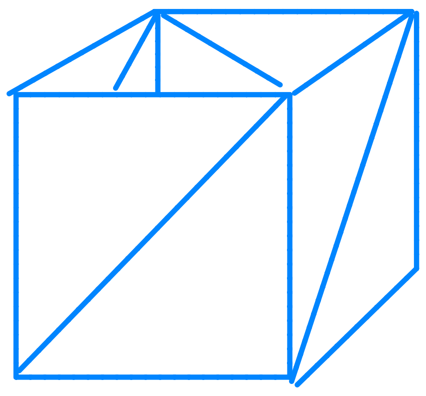
Ex: Icosahedron

$V=12$

$E=30$ $\qquad \chi=2$

$F=20$

Ex: open box (delete top face from cube)

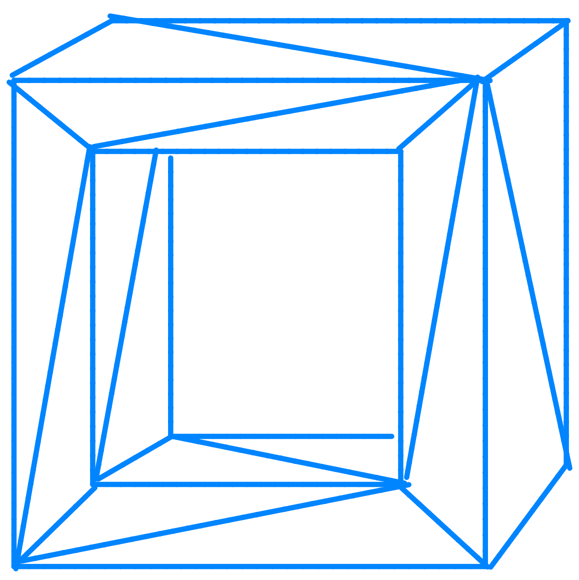


$V=8$

$E=17$ $\qquad \chi=1$

$F=10$

Ex: Picture frame

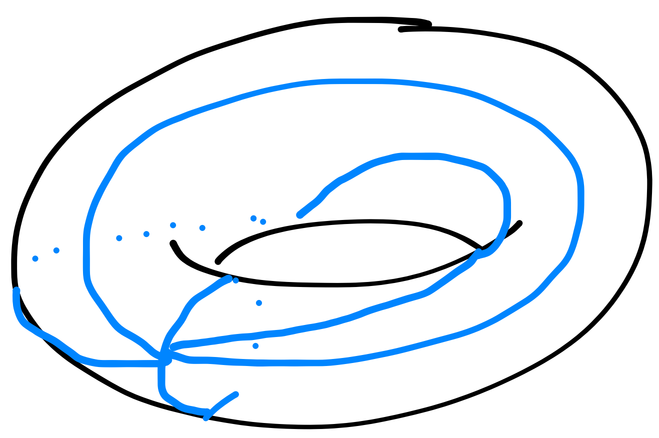


$V=16$

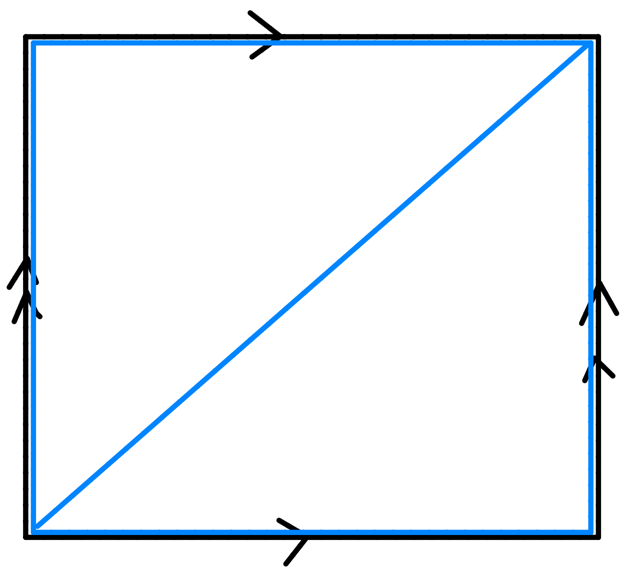
$E=48$ $\qquad \chi=0$

$F=32$

Ex: Torus



(Perhaps better seen in the square, where opposite sides are identified



$V=1$

$E=3$ $\qquad \chi=0$

$F=2$

Amazing facts:

① The Euler characteristic doesn't depend on your choice of triangulation... it really only depends on the underlying surface.

② It doesn't really depend much on the geometry of the surface... it is invariant (unchanged) under bending & stretching (but not tearing)

In fact, as one often proves in a first-semester topology course

$\chi = 2$ $\qquad\qquad$ $\chi = 0$ $\qquad\qquad$ $\chi = -2$

In general, for compact, oriented surfaces w/o bdy, $\chi = 2-2g$, where g is the genus of the surface, which counts the # of handles (or holes)

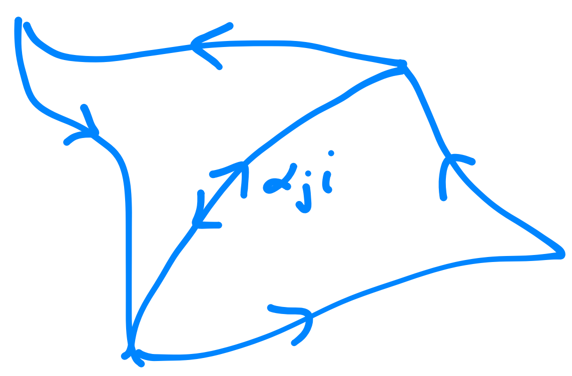
Proof of Global Gauss-Bonnet: We know each triangle adds 3 edges, but each edge is shared by 2 different triangles, so

$$3F = 2E.$$

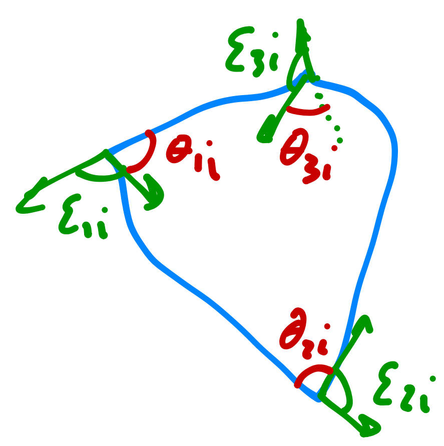
By local Gauss-Bonnet,

$$\iint_\Sigma K\,dArea = \sum_{\Delta_i \in T} \iint_{\Delta_i} K\,dArea = \sum_{\Delta_i \in T}\left[2\pi - \sum_{j=1}^{3}\varepsilon_{ji} - \sum_{j=1}^{3}\int_{\alpha_{ji}}\kappa_g(s)\,ds\right]$$

Now, along each edge α_{ji}, the geodesic curvature integral appears twice w/ opposite signs, so the geodesic curvature terms net out to 0.



Now, at each vertex write $\varepsilon_{ji} = \pi - \theta_{ji}$



So we have $2\pi - \sum_{j=1}^{3}\varepsilon_{ji} = \sum_{j=1}^{3}\theta_{ji} - \pi$, so the above reduces to

$$\iint_\Sigma K\,dArea = \sum_i\left[\sum_{j=1}^{3}\theta_{ji} - \pi\right]$$

But now, if we think about summing over vertices instead of faces, the angles incident to a single vertex must sum to 2π, So $\sum_i \sum_j \theta_{ji} = 2\pi V$. Since $\sum_{\Delta_i \in T} \pi = \pi F$, we have

$$\iint_\Sigma K\,dArea = 2\pi V - \pi F = 2\pi V - \pi F + \pi(3F-2E) = 2\pi(V-E+F) = 2\pi\chi_T(\Sigma).$$ ∎

In fact, this proves amazing fact ①, since $\iint_\Sigma K\,dArea$ is obviously independent of the choice of triangulation.

In fact, by being a little bit more careful (you have to keep track of & treat differently interior vertices/edges & boundary vertices/edges) we can show (see Shifrin)

Global Gauss-Bonnet: If Σ is a compact, orientable surface w/ piecewise-smooth boundary,

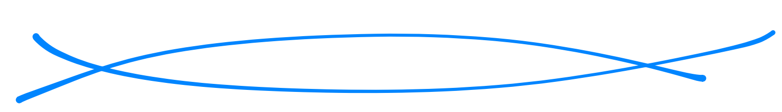
$$\int_{\partial\Sigma} \kappa_g(s)\,ds + \iint_\Sigma K\,dArea + \sum \varepsilon_i = 2\pi\chi(\Sigma)$$

where the ε_i are the exterior angles at corners of $\partial\Sigma$.

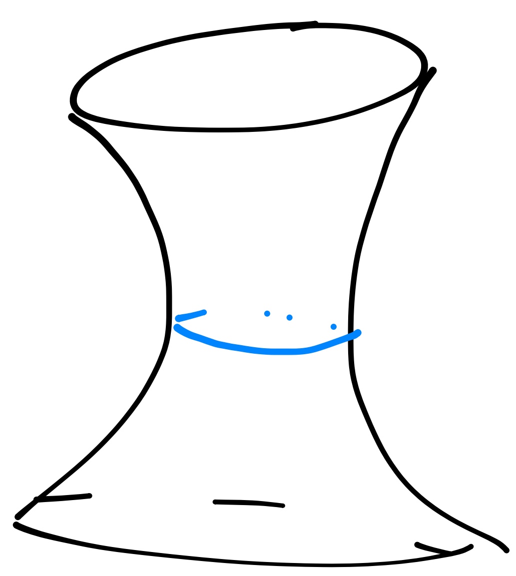
Some consequences:

① A compact surface w/o boundary of positive Gaussian curvature is (topologically) a sphere.

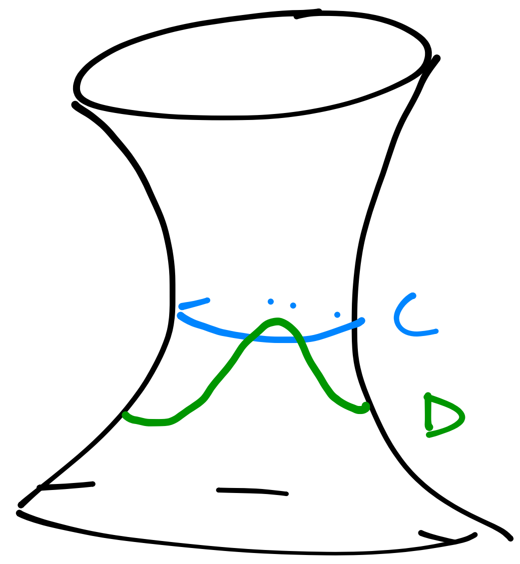
② As already stated, geodesics in negatively curved surfaces cannot meet twice so that the region b/w the intersections is simply connected



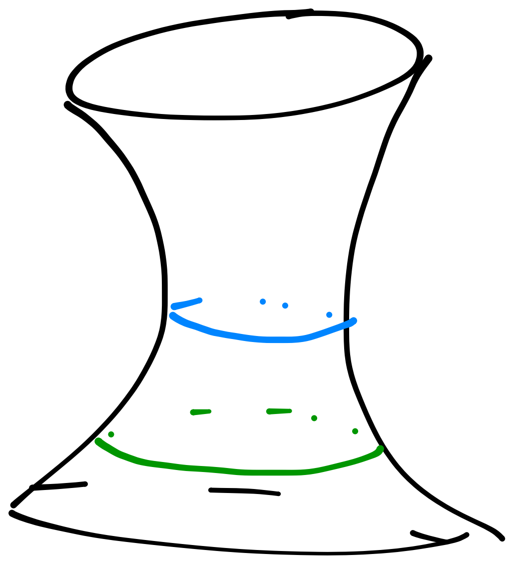
③ Let Σ be a topological cylinder w/ negative curvature (for example, the catenoid). Then Σ has at most one simple closed geodesic



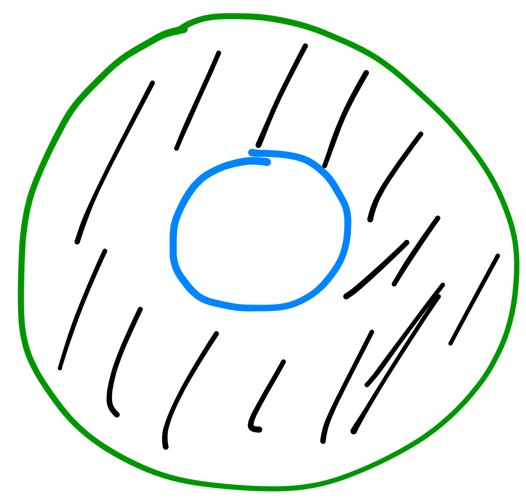
Proof: Suppose there were 2, called C & D. If $C \cap D \neq \emptyset$, then consecutive intersections bound a simply connected region, which is impossible by ②.



So then we must have

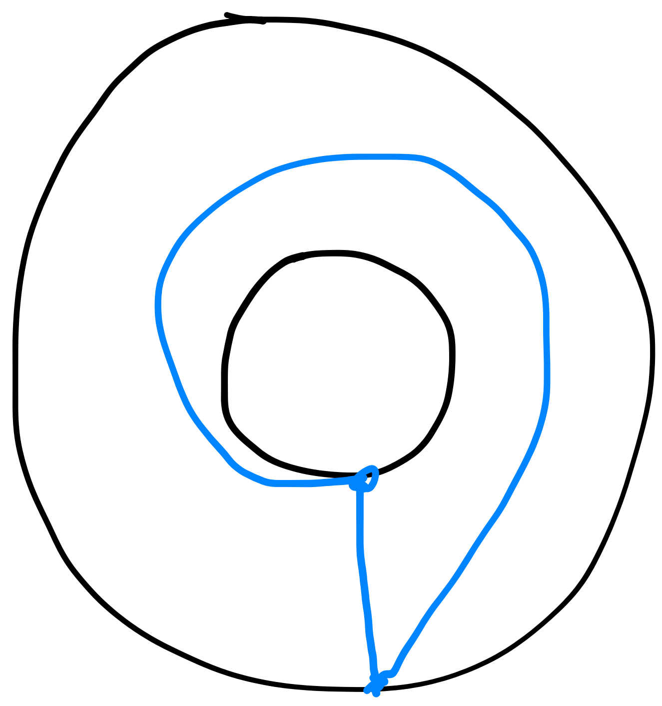


But then the region b/w C & D is an annulus:



Claim: $\chi(\circledcirc) = 0$

Pf v1: Since χ is independent of triangulation, we just have to find a triangulation for some annulus & compute χ of that triangulation.



$V = 2$

$E = 4$

$F = 2$

$\chi = 0$

$\blacksquare$

Pf v2: By Gauss-Bonnet, $\iint_A K\,dArea + \int_{\partial A} \kappa_g\,ds = 2\pi\chi(A)$

For the annulus in the plane, $K=0$ & the total curvatures of the inner & outer circles are both $\pm 2\pi$. In fact, the 2 circles have to be oppositely oriented, so the geodesic curvature terms cancel & we get $0 = 2\pi\chi(A) \Rightarrow \chi(A) = 0$. $\blacksquare$

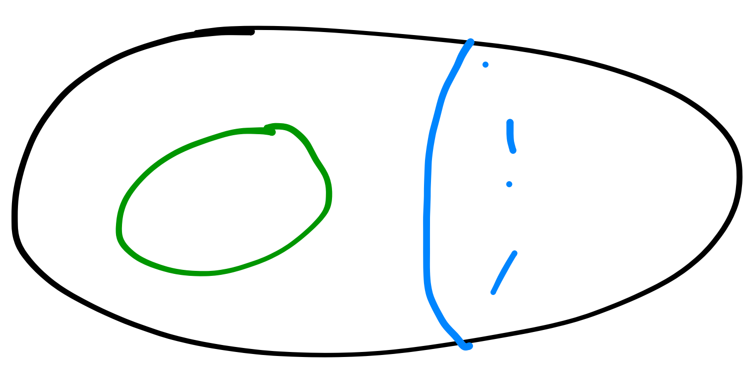
So then, on our negatively-curved cylinder, we have the region b/w C & D having $\chi = 0$, so, by Gauss-Bonnet,

$$\iint K\,dArea = 0$$

which is impossible since $K < 0$. $\blacksquare$

④ Any two simple closed geodesics on a surface of positive curvature must intersect.

Pf: Suppose they didn't. By ①, the surface is topologically a sphere, so the region b/w the curves must be an annulus.



But then $\iint K\,dArea = 2\pi\chi = 0$, which is impossible since $K > 0$. $\blacksquare$

As we'll start discussing next week, there's no real need to assume surfaces are contained in $\mathbb{R}^3$.

So long as you know the first fundamental form, you can compute the Christoffel symbols &, hence, the Gaussian curvature & geodesics.

Next week we'll talk about the hyperbolic plane, which is the fundamental surface w/ constant negative curvature.