Math 474: Days 35-36

Now that we've seen geodesic curvature is important, is there a way to compute it intrinsically?

Well, suppose $\alpha'(s) = u'(s)\vec{X}_u + v'(s)\vec{X}_v$. Then

$$\alpha''(s) = u''(s)\vec{X}_u + u'(s)\frac{d}{ds}\vec{X}_u + v''(s)\vec{X}_v + v'(s)\frac{d}{ds}\vec{X}_v.$$

Now,

$$\frac{d}{ds}\vec{X}_u(u(s),v(s)) = u'(s)\vec{X}_{uu} + v'(s)\vec{X}_{uv}$$

$$\frac{d}{ds}\vec{X}_v(u(s),v(s)) = u'(s)\vec{X}_{vu} + v'(s)\vec{X}_{vv}$$

So then, writing $\vec{X}_{uu}$ & $\vec{X}_{uv}$ in terms of the basis $\{\vec{X}_u, \vec{X}_v, \vec{N}\}$ gives

$$\alpha''(s) = \left(u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1\right)\vec{X}_u + \left(v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2\right)\vec{X}_v + (\text{stuff})\vec{N}$$

Since the tangential part gives the geodesic curvature, the curve is a geodesic $\Leftrightarrow$ it satisfies the following geodesic equations:

$$u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1 = 0$$

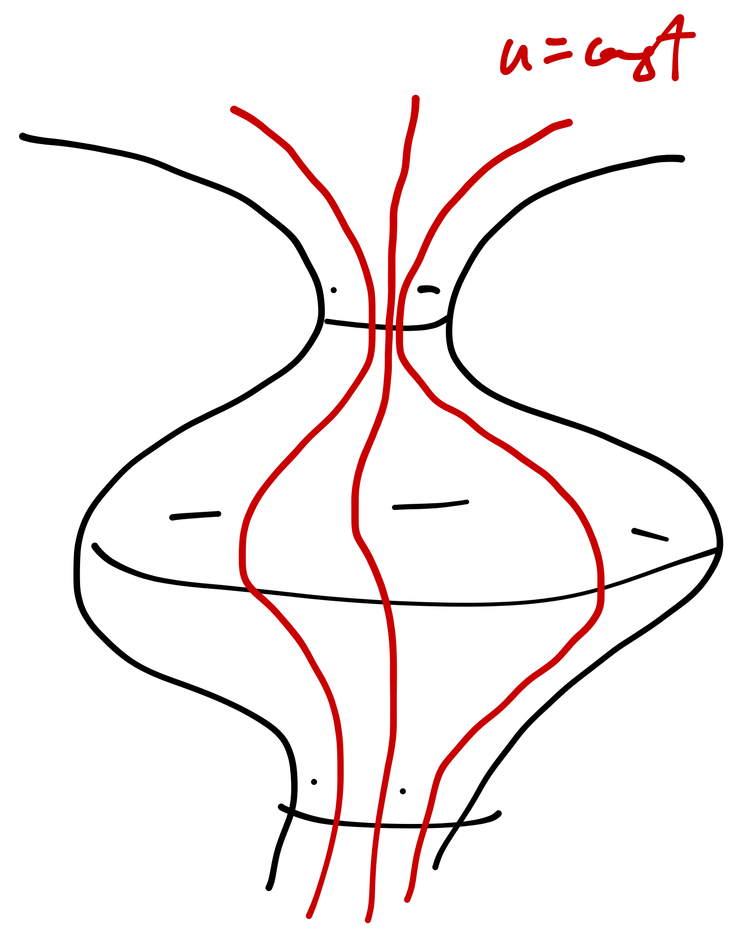
$$v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^2 = 0$$

Ex: Surfaces of revolution. If Σ parametrized by

$\vec{x}(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$, then we've seen before that

$\Gamma_{11}^1 = 0, \quad \Gamma_{11}^2 = -\dfrac{\varphi\varphi'}{(\varphi')^2+(\psi')^2}$. Also, by similar reasoning,

$$\Gamma_{12}^1 = \frac{\varphi'}{\varphi}, \quad \Gamma_{12}^2 = 0, \quad \Gamma_{22}^1 = 0, \quad \Gamma_{22}^2 = \frac{\varphi'\varphi'' + \psi'\psi''}{(\varphi')^2+(\psi')^2}$$



So the geodesic eqns become

$$u'' + 2\frac{\varphi'}{\varphi}u'v' = 0$$

$$v'' - (u')^2\frac{\varphi\varphi'}{(\varphi')^2+(\psi')^2} + (v')^2\frac{\varphi'\varphi''+\psi'\psi''}{(\varphi')^2+(\psi')^2} = 0$$

Claim: Meridians ($u=\text{const}$, $v=v(s)$) are always geodesics

Pf: Since $u=\text{const}$, $u'=0=u''$, so the first eqn is satisfied & the second becomes

$$v'' + (v')^2\frac{\varphi'\varphi''+\psi'\psi''}{(\varphi')^2+(\psi')^2} = 0$$

Now, since $(u, v(s))$ is parametrized by arclength (it's just a rotated copy of the original curve used to generate the surface)

we know that $\frac{d}{ds}(u, v(s)) = v'(s)\vec{x}_v$ has unit norm, so

$$1 = \langle v'(s)\vec{x}_v, v'(s)\vec{x}_v\rangle = (v')^2 G = (v')^2\cdot((\varphi')^2+(\psi')^2),$$

meaning that $(v')^2 = \frac{1}{(\varphi')^2+(\psi')^2}$. Now, I want to differentiate this, but I have to be a little careful w/ my notation.

Here $v' = v'(s)$, but $\varphi' = \varphi'(v)$ & $\psi' = \psi'(v)$, so I have

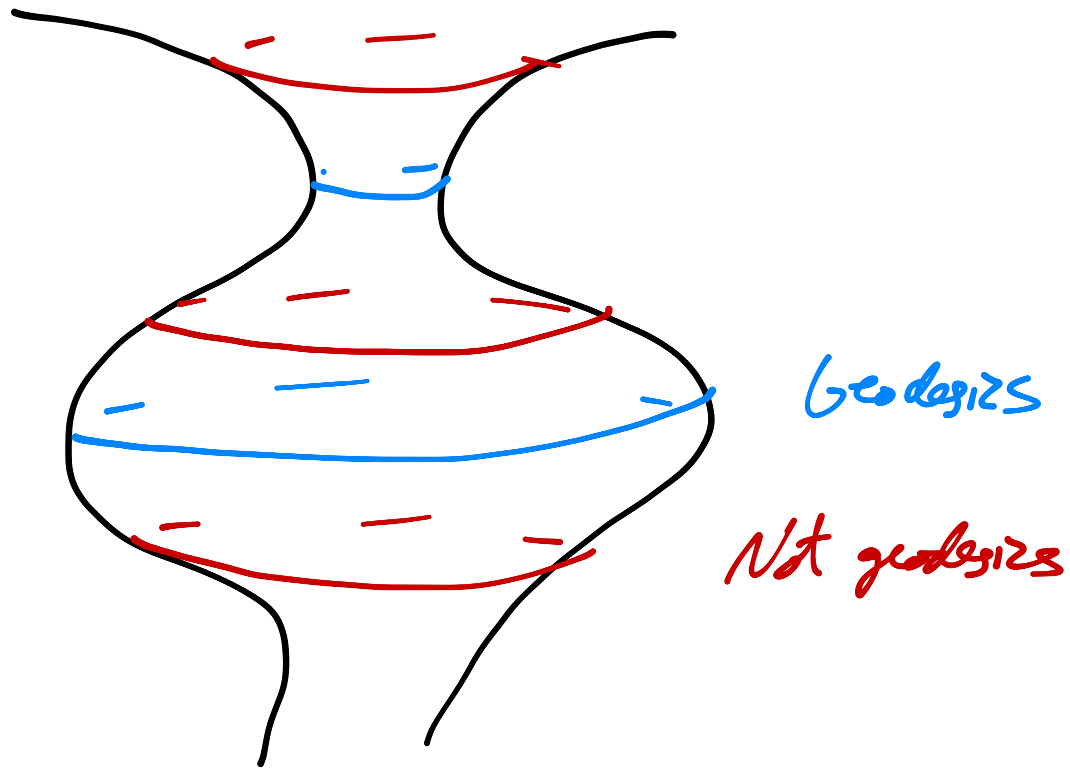
$$\frac{d}{ds}(v'(s)^2) = 2v'(s)v''(s) = \frac{d}{ds}\left(\frac{1}{(\varphi'(v))^2+(\psi'(v))^2}\right) = \frac{-1}{((\varphi'(v))^2+(\psi'(v))^2)^2}\cdot(2\varphi'(v)\varphi''(v)+2\psi'(v)\psi''(v))\cdot\frac{d}{ds}v(s)$$

So

$$\cancel{2}\cancel{v'}v'' = -\cancel{2}\,\frac{\varphi'\varphi''+\psi'\psi''}{((\varphi')^2+(\psi')^2)^2}\,\cancel{v'}$$

$$v'' = -\frac{\varphi'\varphi''+\psi'\psi''}{((\varphi')^2+(\psi')^2)^2} = -\frac{\varphi'\varphi''+\psi'\psi''}{(\varphi')^2+(\psi')^2}(v')^2,$$ as desired.

Claim: Parallels ($u = u(s)$, $v = \text{const}$) are geodesics $\iff$ $\varphi' = 0$ on the parallel



Pf: The first eqn becomes $u'' = 0$, so $u' = \text{const}$.

The second is

$$(u')^2\frac{\varphi\varphi'}{(\varphi')^2+(\psi')^2} = 0$$

So either

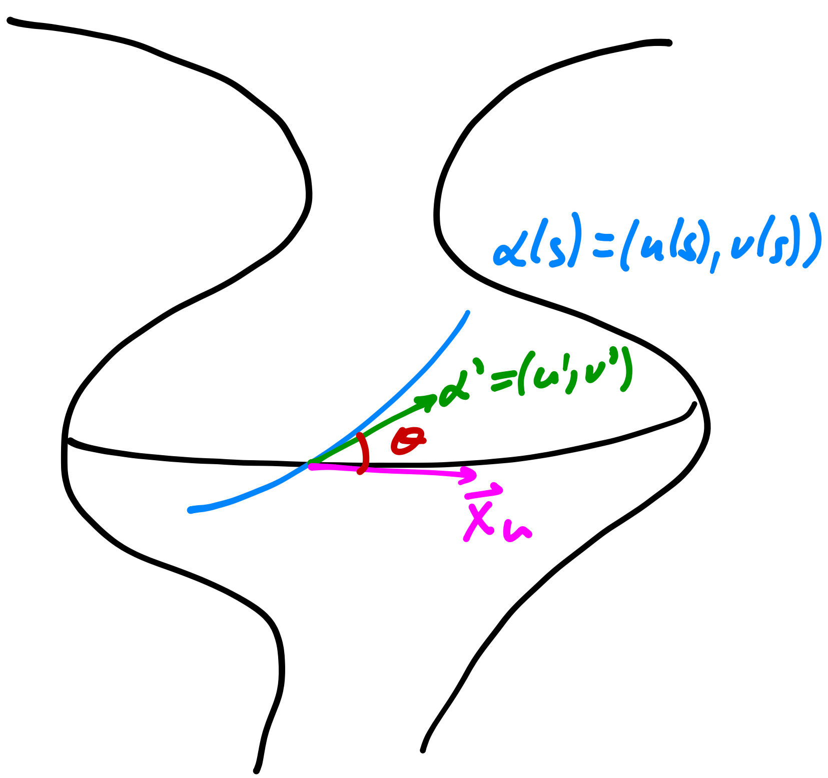
- $u' = 0$: Since $u' = \text{const}$, this implies $u' \equiv 0$, so $(u(s), c) = (c_1, c_2)$ is a point.
- $\varphi = 0$: Σ not regular (it's just a portion of the z-axis)
- $\varphi' = 0$: So this must be the case!

What about the other geodesics?

Consider $u'' + 2\frac{\varphi'}{\varphi}u'v' = 0 \iff \varphi^2 u'' + 2\varphi\varphi' u'v' = 0 \iff \frac{d}{ds}(\varphi(v)^2 u'(s)) = \varphi^2 u'' + 2\varphi\varphi' v'u' = 0$

$\iff \varphi^2 u' = \text{const}$

What does this mean geometrically?



So $\cos\theta = \dfrac{\langle u'\vec{x}_u + v'\vec{x}_v, \vec{x}_u\rangle}{|u'\vec{x}_u + v'\vec{x}_v|\,|\vec{x}_u|}$

but $|u'\vec{x}_u + v'\vec{x}_v| = |\alpha'| = 1$ since α is arclength-parametrized.

Also, $|\vec{x}_u| = \sqrt{E}$ & $\langle u'\vec{x}_u + v'\vec{x}_v, \vec{x}_u \rangle = u'E + v'F = u'\varphi^2$, so

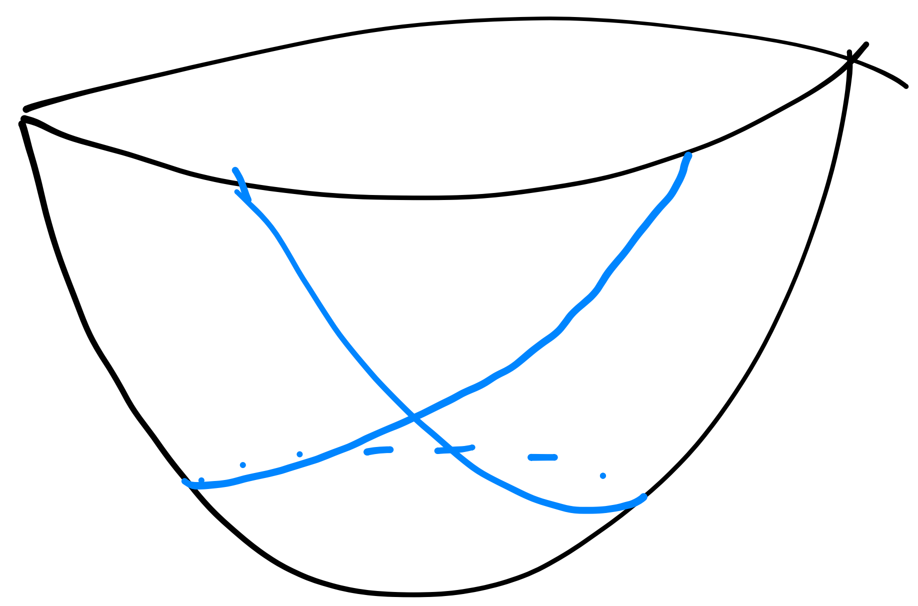
$$\cos\theta = \varphi u' \quad \& \text{ hence } \quad \varphi^2 u' = \varphi\cos\theta = \text{const}$$

This is Clairaut's relation for geodesics on surfaces of revolution, which says

$$r\cos\theta = \text{const} \quad \text{for geodesics.}$$

In particular, a geodesic can never be tangent to a meridian unless it is a meridian, so geodesics always spiral clockwise or counterclockwise

Ex: On a paraboloid, any non-meridian geodesic intersects itself infinitely many times.



Ex: (Mathematica demo) geodesics on the torus.