Math 474: Days 2-3

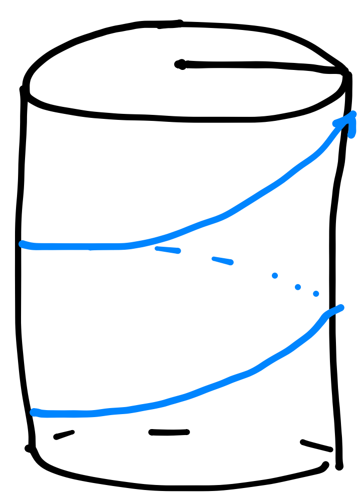
Curve Theory:

Def: A smooth parametrized curve in $\mathbb{R}^3$ (or $\mathbb{R}^2$, or generally $\mathbb{R}^n$) is a smooth (i.e., infinitely differentiable) map $\alpha:(a,b)\to\mathbb{R}^3$ where $a,b\in\mathbb{R}$.

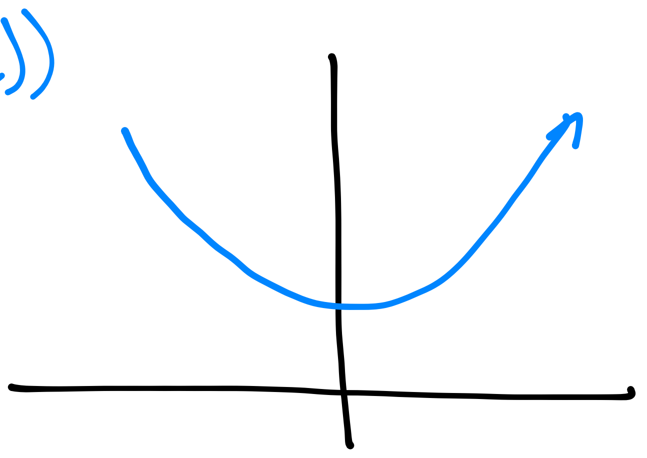
We usually write $\alpha(t)=(x(t),y(t),z(t))$ & the velocity vector or tangent vector is $\alpha'(t)=(x'(t),y'(t),z'(t))$.

Ex: $\alpha(t)=(a\cos t, a\sin t, bt)$

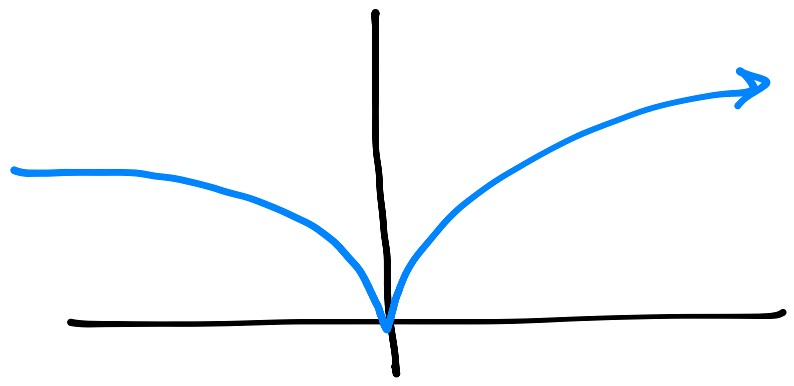
The image of $\alpha(t)$ is a helix



Ex: The catenary $(t, C\cosh(t/c))$



Ex: The cubic $\alpha(t)=(t^3,t^2)$



We usually want to exclude this last example by assuming regularity:

Def: A smooth parametrized curve $\alpha(t)$ is regular if $\alpha'(t)\neq\vec{0}$ for all t in the domain.

Def: The arclength of a curve $\alpha(t)$ b/w t_0 & t is defined to be

$$s(t)=\int_{t_0}^{t}|\alpha'(u)|\,du$$

If $|\alpha'(u)|=1$ $\forall u$, then we observe that $s(t)=t-t_0$ & we say α is parametrized by arclength.

It turns out to be theoretically convenient to assume curves are parametrized by arclength, so we'll usually make that assumption when proving theorems. However, it's computationally impractical to get curves to be parametrized by arclength, so we'll also need to be able to compute w/ other parametrizations.

Now, to start talking about curvature, we need to think about what curvature should mean.

How can we characterize a straight line in terms of its parametrization?

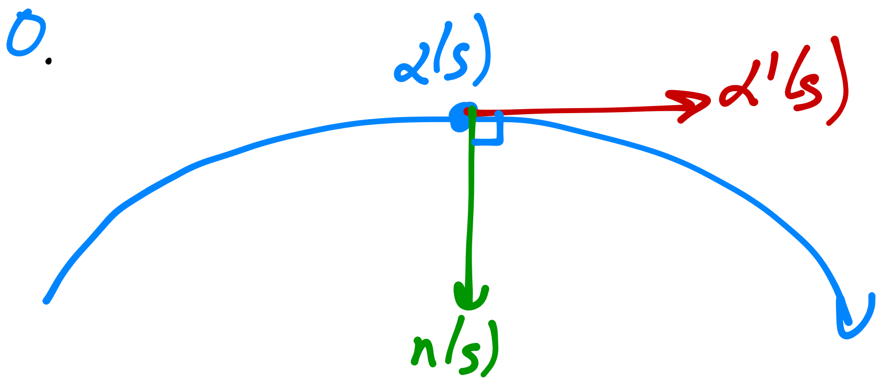
Well, if the line is parametrized by $\alpha(t)$, the the tangent vectors $\alpha'(t)$ should all be pointing in the same direction.

Whereas if $\alpha(t)$ is not a straight line, then the tangent vectors $\alpha'(t)$ should be changing; of course, $\alpha''(t)$ is telling us how the tangent vector is changing.

Def: Let $\alpha: I \to \mathbb{R}^3$ be a curve parametrized by arclength. The $|\alpha''(s)|$ is called the curvature at $\alpha(s)$, usually denoted $\kappa(s)$

Now, of course $\alpha''(s) = |\alpha''(s)| \cdot \frac{\alpha''(s)}{|\alpha''(s)|} = \kappa(s)\, \vec{n}(s)$ where $\vec{n}(s) = \frac{\alpha''(s)}{|\alpha''(s)|}$ is a unit vector, called the standard normal.

Obviously, $\vec{n}(s)$ is only well-defined if $\kappa(s) \neq 0$.



Prop: $\vec{n}(s)$ is perpendicular to $\alpha'(s)$.

Proof: It suffices to show that $\vec{n}(s) \cdot \alpha'(s) = 0$. To prove this, we'll differentiate $\alpha'(s) \cdot \alpha'(s) = 1$:

$$0 = \frac{d}{ds}(\alpha'(s)\cdot\alpha'(s)) = \alpha''(s)\cdot\alpha'(s) + \alpha'(s)\cdot\alpha''(s) = 2\alpha'(s)\cdot\alpha''(s) = 2\alpha'(s)\cdot(\kappa(s)\vec{n}(s)) = 2\kappa(s)(\alpha'(s)\cdot\vec{n}(s)).$$

So long as $\kappa(s) \neq 0$, this means that $\vec{t}(s) := \alpha'(s)$ & $\vec{n}(s)$ give an orthonormal basis for a plane containing $\alpha(s)$, called the osculating plane.

Of course, given an (oriented) plane, there's a unique perpendicular vector. In this case we call that vector $\vec{b}(s)$ the binormal vector; of course $\vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$.

Now, the derivative of the binormal is telling us the rate at which $\alpha(s)$ is pulling out of the osculating plane.

So compute: $\vec{b}'(s) = \frac{d}{ds}(\vec{t}(s)\times\vec{n}(s)) = \vec{t}'(s)\times\vec{n}(s) + \vec{t}(s)\times\vec{n}'(s) = \vec{t}(s)\times\vec{n}'(s)$ since $\vec{t}'(s) = \kappa(s)\,\vec{n}(s)$.

Hence, $\vec{b}'(s)\cdot\vec{b}(s) = 0$ (since $\vec{b}(s)$ is unit)

$\vec{b}'(s)\cdot\vec{t}(s) = (\vec{t}(s)\times\vec{n}(s))\cdot\vec{t}(s) = 0$,

So it must be the case that $\vec{b}(s)$ is a scalar multiple of $\vec{n}(s)$.

Def: The torsion of $\alpha(s)$ is the quantity $\tau(s)$ given by

$$\vec{b}'(s) = \tau(s)\,\vec{n}(s).$$

Now obviously plane curves have zero torsion (since the binormal is constant) & any curve w/ nonvanishing curvature & vanishing torsion must be a plane curve (can you think of a counterexample when curvature is allowed to vanish?)

The triple $(\vec{t}(s), \vec{n}(s), \vec{b}(s))$ is an ON basis (or frame) at each point of $\alpha(s)$, called the Frenet frame. Some facts:

$$\begin{aligned}
\vec{t}\,'(s) &= \kappa(s)\,\vec{n}(s) \\
\vec{n}\,'(s) &= -\kappa(s)\,\vec{t}(s) - \tau(s)\,\vec{b}(s) \\
\vec{b}\,'(s) &= \tau(s)\,\vec{n}(s)
\end{aligned}$$

$$\begin{aligned}
\vec{n}\,'(s) = \tfrac{d}{ds}\big(\vec{b}(s)\times\vec{t}(s)\big) &= \vec{b}\,'(s)\times\vec{t}(s) + \vec{b}(s)\times\vec{t}\,'(s) \\
&= \tau(s)\,\vec{n}(s)\times\vec{t}(s) + \vec{b}(s)\times\big(\kappa(s)\,\vec{n}(s)\big) \\
&= -\tau(s)\,\vec{b}(s) - \kappa(s)\,\vec{t}(s)
\end{aligned}$$

It's common to write this as a matrix differential equation: let $F(s) = \begin{bmatrix} — & \vec{t}(s) & — \\ — & \vec{n}(s) & — \\ — & \vec{b}(s) & — \end{bmatrix}$ (which is an orthogonal matrix).

Then $F'(s) = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} F(s)$ (In fact, the analogous statement is true when F is any orthogonal 3×3 matrix!)

$\uparrow$ skew-symmetric

Now, the somewhat amazing fact is that bending ($\kappa(s)$) & twisting ($\tau(s)$) should encode all possible deformations of curves.

Fundamental Theorem of Local Theory of Curves: Given smooth functions $\kappa(s) > 0$ & $\tau(s)$ defined on an interval I, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ so that:

(i) s is the arclength of α

(ii) $\kappa(s)$ is the curvature of α

(iii) $\tau(s)$ is the torsion of α.

Further, any other curve $\bar{\alpha}(s)$ w/ the same curvature & torsion is related to $\alpha(s)$ by a rigid motion (i.e., the traces of α & $\bar{\alpha}$ are congruent).

The proof of existence follows from the standard theory of ODEs (see Appendix 3 for more details). We will prove uniqueness:

Proof of Uniqueness: Arclength, curvature, & torsion are all obviously invariant under rigid motions.

Now, suppose $\alpha(s)$ & $\bar{\alpha}(s)$ have the same $\kappa(s)$ & $\tau(s)$. After applying a rigid motion, we can arrange that

$$\begin{aligned}
\alpha(0) &= \bar{\alpha}(0) \\
t(0) &= \bar{t}(0) \\
n(0) &= \bar{n}(0) \\
b(0) &= \bar{b}(0)
\end{aligned}$$

So now consider a sort of distance b/w frames

$$d(s) := |t(s) - \bar{t}(s)|^2 + |n(s) - \bar{n}(s)|^2 + |b(s) - \bar{b}(s)|^2.$$

Then
$$\begin{aligned}
d'(s) &= \tfrac{d}{ds}\Big(\big(t(s)-\bar{t}(s)\big)\cdot\big(t(s)-\bar{t}(s)\big)\Big) + \tfrac{d}{ds}\Big(\big(n(s)-\bar{n}(s)\big)\cdot\big(n(s)-\bar{n}(s)\big)\Big) + \tfrac{d}{ds}\Big(\big(b(s)-\bar{b}(s)\big)\cdot\big(b(s)-\bar{b}(s)\big)\Big) \\
&= 2\big(t'(s)-\bar{t}\,'(s)\big)\cdot\big(t(s)-\bar{t}(s)\big) + 2\big(n'(s)-\bar{n}'(s)\big)\cdot\big(n(s)-\bar{n}(s)\big) + 2\big(b'(s)-\bar{b}'(s)\big)\big(b(s)-\bar{b}(s)\big)
\end{aligned}$$

Using the Frenet equations, this is

$$= 2\kappa(s)(n(s)-\bar{n}(s))\cdot(t(s)-\bar{t}(s)) - 2\kappa(s)(t(s)-\bar{t}(s))\cdot(n(s)-\bar{n}(s)) - 2\tau(s)(b(s)-\bar{b}(s))\cdot(n(s)-\bar{n}(s))$$

$$+2\tau(s)(n(s)-\bar{n}(s))\cdot(b(s)-\bar{b}(s))$$

$$= 0,$$

So $d(s) = \text{const} = d(0) = 0 \Rightarrow (t(s), n(s), b(s)) = (\bar{t}(s), \bar{n}(s), \bar{b}(s))$. In particular,

$$\alpha'(s) = t(s) = \bar{t}(s) = \bar{\alpha}'(s) \Leftrightarrow \frac{d}{ds}(\alpha(s)-\bar{\alpha}(s)) = 0,$$

meaning that

$$\alpha(s) = \bar{\alpha}(s) + \vec{c}.$$

But since $\alpha(0) = \bar{\alpha}(0)$, this can only be true if $\vec{c} = \vec{0}$, completing the proof. ▨

Now, we've been assuming our curves are parametrized by arclength, but let's show that isn't a restrictive assumption:

Prop: Given a regular smooth curve $\alpha: I \to \mathbb{R}^3$, there exists $\beta: J \to \mathbb{R}^3$ parametrized by arclength with the same trace as α.

Proof: Let $s(t) = \int_{t_0}^{t} |\alpha'(t)|\,dt$. Then since α is regular, $|s'(t)| = |\alpha'(t)| > 0$, so the function s has a differentiable inverse $F(s)$ by the Inverse Function Theorem.

> **Inverse Function Thm:** If $g: I \to \mathbb{R}$ is continuously differentiable with nonzero derivative on the domain I, then g is invertible on I, the inverse function g^{-1} is continuously differentiable, and $(g^{-1})'(g(a)) = \frac{1}{g'(a)}$.
>
> More generally, if $G: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable & $\det(DG) \neq 0$ on U, then G is invertible on U
>
> & $DG^{-1}(G(\vec{p})) = [DG(p)]^{-1}$.

Since $F(s(t)) = t$, we know $1 = F'(s(t))\, s'(t)$.

Now, if we define β by $\beta(s) = \alpha(F(s))$, then

$$|\beta'(s)| = \left|\frac{d}{ds}(\alpha(F(s)))\right| = |\alpha'(F(s))| \cdot F'(s) = s'(F(s))\,F'(s) = 1,$$ so β is parametrized by arclength.

Def: If $\alpha(t)$ is any regular parametrized curve, we define the curvature & torsion for $\alpha(t)$ at t by:

- $\kappa(t)$ = the curvature of the reparametrization of α by arclength at t.
- $\tau(t)$ = the torsion of the reparametrization of α by arclength at t.