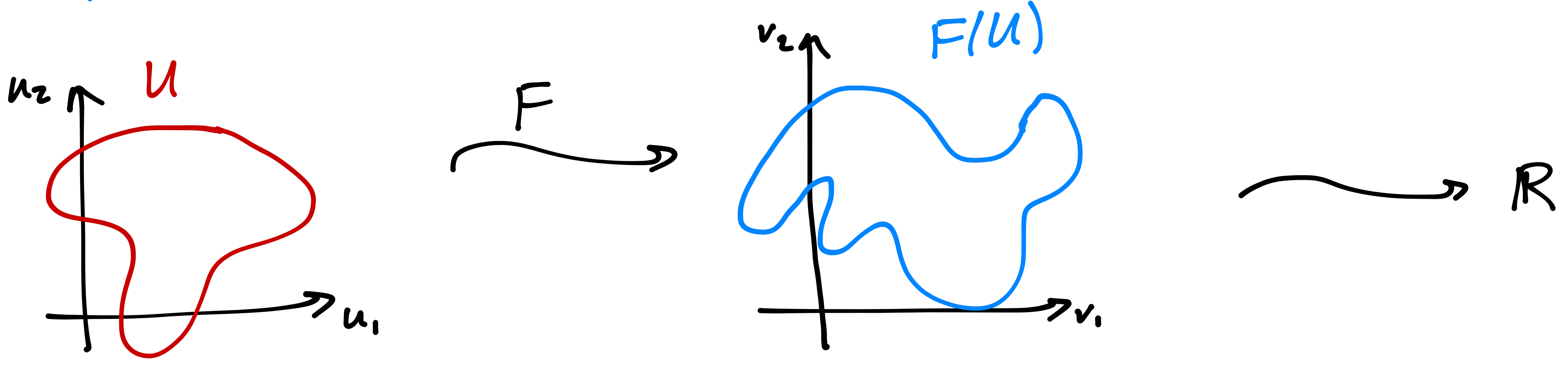
Math 474: Day 9

I want to build up to another integral geometric formula, but since everything depends on the change of variables formula from multivariable calculus, first some review.

Suppose $F: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one & has continuously differentiable partials with $|DF| \neq 0$. Then for any continuous φ,

$$\int_{F(U)} \varphi(\vec{v})\,d\vec{v} = \int_{U} \varphi(F(\vec{u}))\,|DF|\,d\vec{u}$$



Ex: $F(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$ is the map from polar to rectangular coordinates. Then

$$\int_{F(U)} \varphi(x,y)\,dx\,dy = \int_{U} \varphi(r,\theta)\,|DF|\,dr\,d\theta$$

But now
$$|DF| = \begin{vmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial \theta} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial \theta}(r\cos\theta) \\ \frac{\partial}{\partial r}(r\sin\theta) & \frac{\partial}{\partial \theta}(r\sin\theta) \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r,$$
so

$$\iint \varphi(x,y)\,dx\,dy = \iint \varphi(r,\theta)\,r\,dr\,d\theta \ldots$$ which you hopefully were expecting.

Df: A map $F: \mathbb{R}^2 \to \mathbb{R}^2$ is a **rigid motion** if it is a combination of a rotation & a translation.

A rotation by θ is the linear map

$$R_\theta(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

A translation by (v_1, v_2) is the map

$$T_{(v_1,v_2)}(x,y) = (x+v_1,\ y+v_2)$$

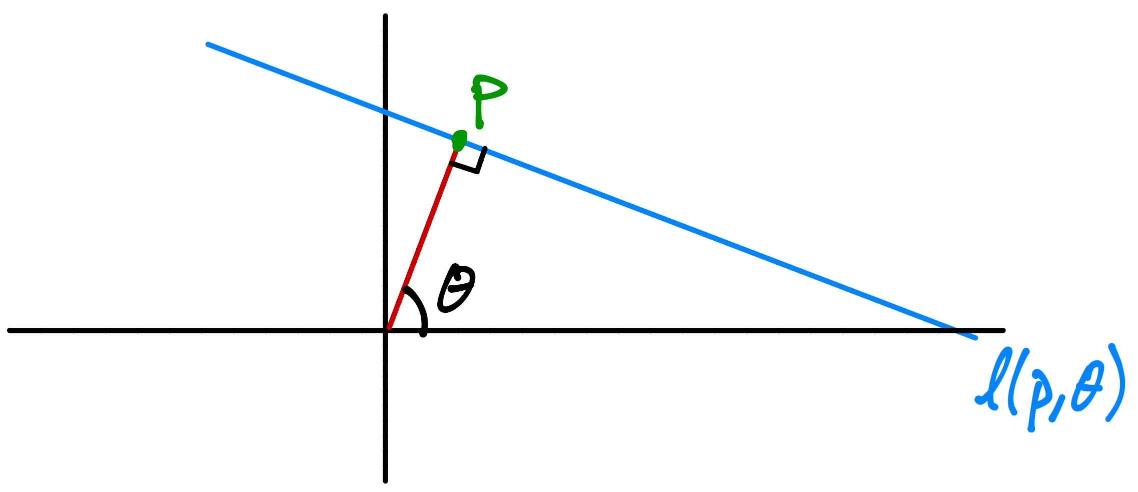
So, combining the two, we see that an arbitrary rigid motion is of the form

$$F_{\theta,(v_1,v_2)}(x,y) = (x\cos\theta - y\sin\theta + v_1,\ x\sin\theta + y\cos\theta + v_2)$$

Df: Let $\mathcal{L}$ be the set of lines in $\mathbb{R}^2$. We can parametrize (or coordinatize) this space by letting $\ell(\theta, p)$ be the line

$$(\cos\theta)x + (\sin\theta)y = p$$

Pictorially,



Okay, so now here's (finally) the theorem:

Thm (Crofton's Formula): If α is a curve in $\mathbb{R}^2$, then

$$\text{Length}(\alpha) = 4 \iint I_\alpha(p,\theta)\, dp\, d\theta$$

where $I_\alpha(p,\theta)$ is the number of intersections of α w/ the line $\ell(p,\theta)$.