# Math 474: Day 8

New paradigm: Integral Geometry.

The idea here is, instead of analyzing local information about curves (e.g., curvature), we average global information about the curve over all possible ways of looking at it.

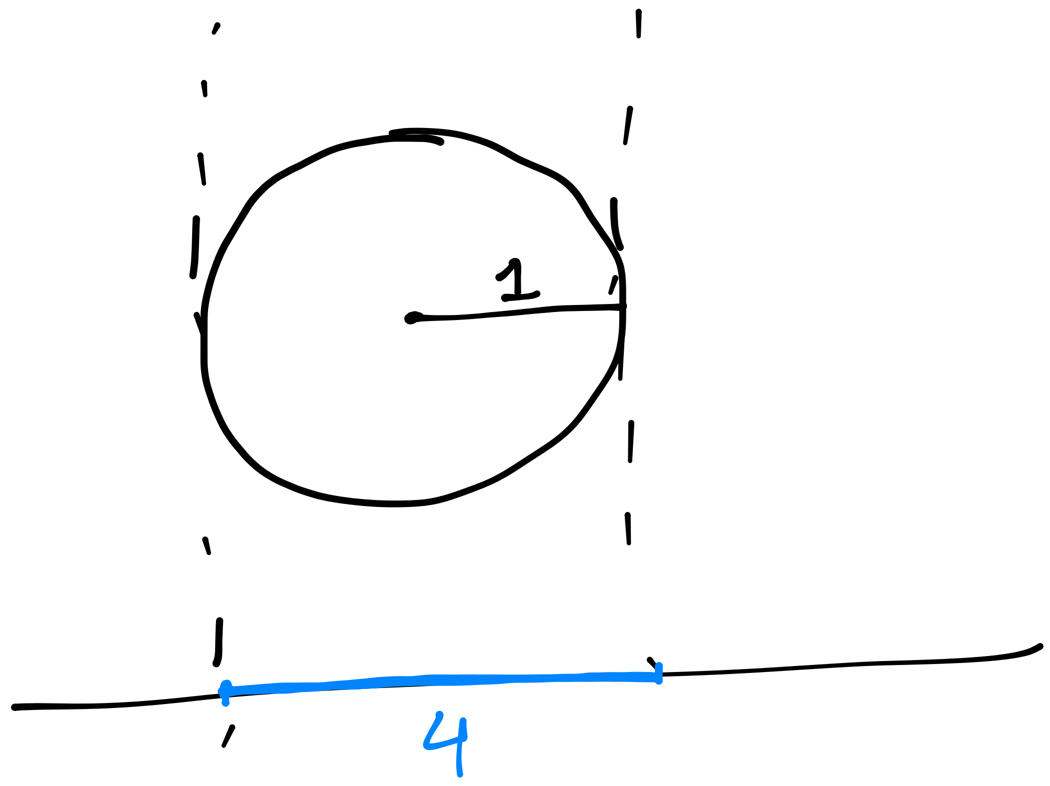
We'll start with a new approach to finding the length of a plane curve.

Thm: Let $\alpha$ be a plane curve. Given a direction $\vec{p} \in S^1$, let $\text{len}_{\vec{p}}\,\alpha$ be the length of the projection of $\alpha$ to the line containing $\vec{p}$. Then

$$\text{Length}(\alpha) = \frac{1}{4}\int_{\vec{p}\in S^1} \text{len}_{\vec{p}}\,\alpha \, d\vec{p}.$$

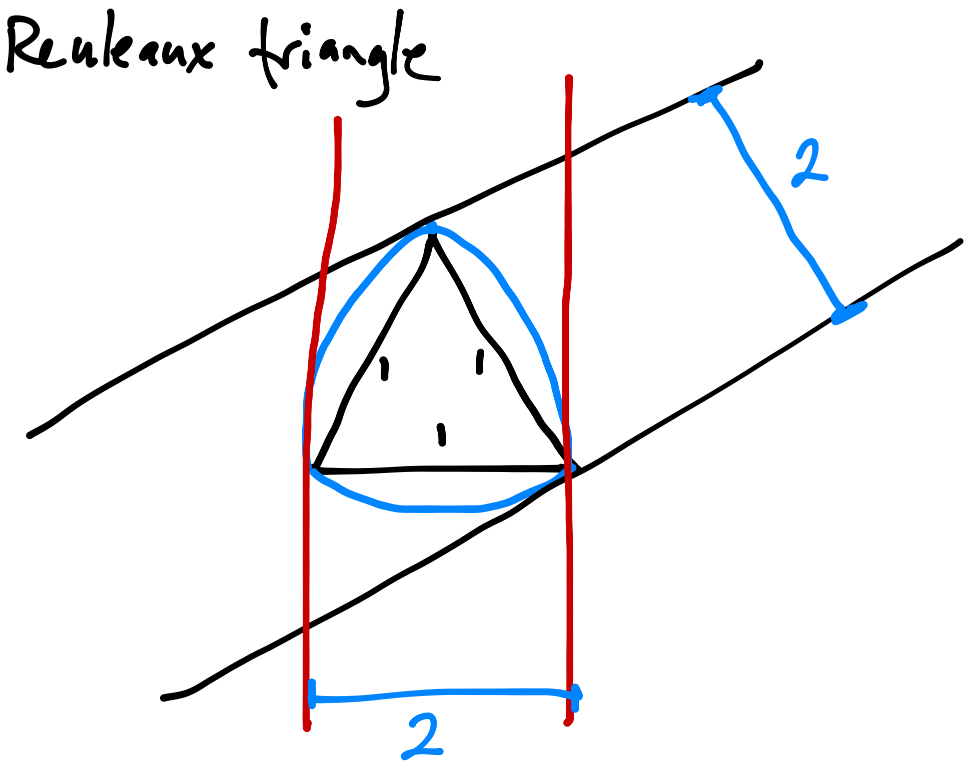
In other words, up to some constant factor the length of a curve is the average length of its projections onto all lines through the origin(!)

Ex:



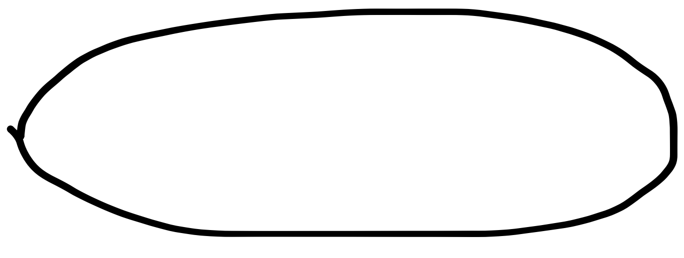

so $\frac{1}{4}\int_0^{2\pi} 4\,d\theta = 2\pi = \text{Length}(\alpha)$ ✓

Ex: Reuleaux triangle




so $\frac{1}{4}\int_0^{2\pi} 2\,d\theta = \pi$ & $\text{Length}(\alpha) = 3\cdot\frac{2\pi}{6} = \pi$. ✓

Ex:



$\alpha(t) = (a\cos t, b\sin t)$

$\vec{p}(\theta) = (\cos\theta, \sin\theta)$

$$\frac{1}{4}\int_0^{2\pi} \text{len}_{\vec{p}(\theta)}\,\alpha \, d\theta = \frac{1}{4}\int_0^{2\pi}\left(\int_0^{2\pi} \alpha'(t)\cdot\vec{p}(\theta)\,dt\right)d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^{2\pi}(-a\sin t\cos\theta + b\cos t\sin\theta)\,dt\,d\theta$$

$$= \dots ?$$

Proof of Thm: We can write the projection of $\alpha$ to $\vec{p}$ as $(\vec{p}\cdot\alpha(s))\vec{p}$, so

$$\text{len}_{\vec{p}}\,\alpha = \int_{s=0}^{\ell} \left|(p\cdot\alpha(s))'\right| ds = \int_0^{\ell} |p\cdot\alpha'(s)|\,ds = \int_0^{\ell} |\vec{p}|\,|\alpha'(s)|\,|\cos\theta|\,ds = \int_0^{\ell} |\cos\theta|\,ds$$

where $\theta$ is the angle b/w $\alpha'(s)$ & $\vec{p}$. Now, as $\vec{p}$ varies around the circle, $\theta$ varies from $0$ to $2\pi$

$$\int_{\theta=0}^{2\pi}\int_{s=0}^{\ell} |\cos\theta|\,ds\,d\theta = \int_0^{\ell}\left(\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta + \int_{\pi/2}^{3\pi/2} -\cos\theta\,d\theta\right) ds = \int_0^{\ell}\left(\sin\theta\Big]_{-\pi/2}^{\pi/2} - \sin\theta\Big]_{\pi/2}^{3\pi/2}\right) ds = \int_0^{\ell} 4\,ds = 4\ell.$$

So then $\frac{1}{4}\int_{\theta=0}^{2\pi} \text{len}_{p(\theta)}\alpha\,d\theta = \ell$, as desired. $\blacksquare$