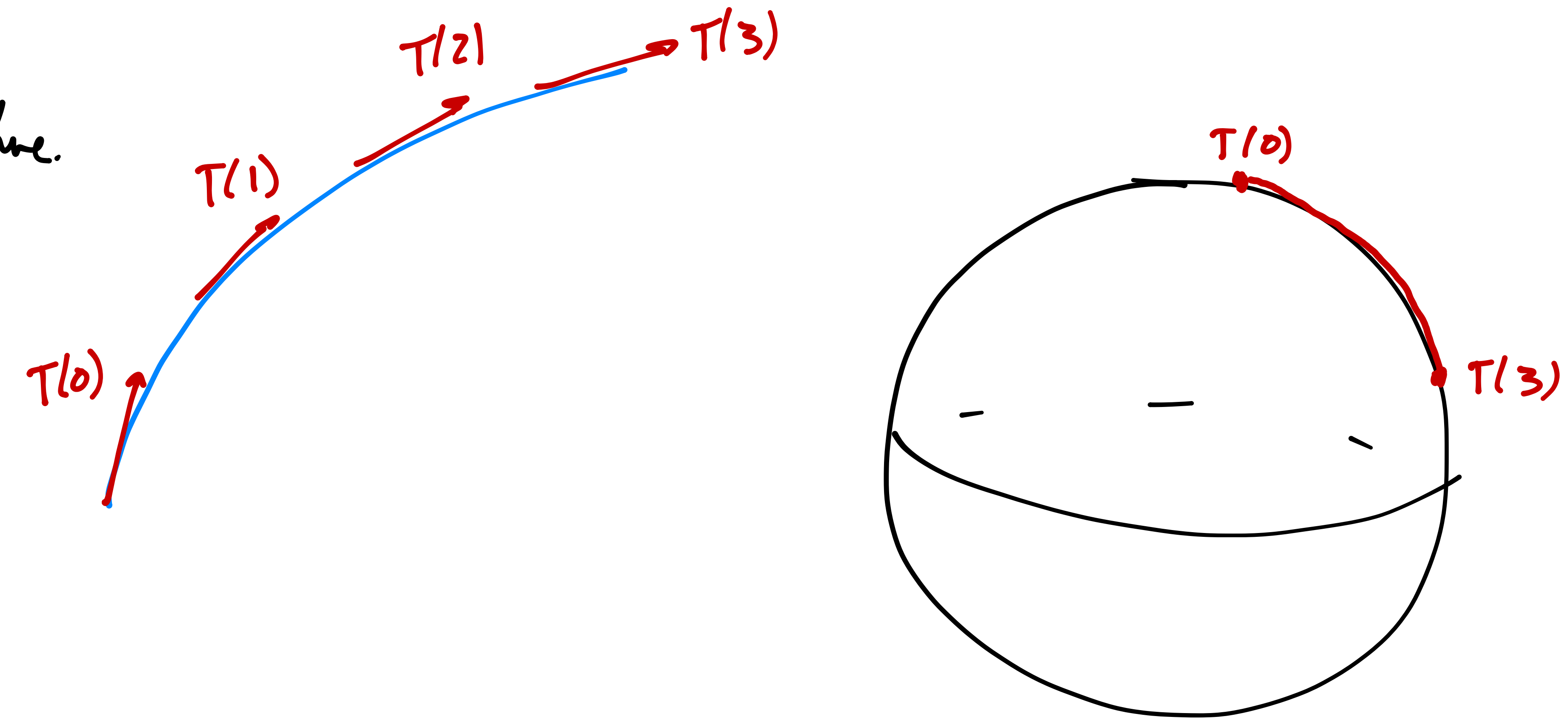
Math 474: Day 7

Shifting gears to plane curves...

Df: The tangent indicatrix of a unit-speed parametrized differentiable curve $\alpha(s)$ is the curve $T(s) = \alpha'(s)$.

Clearly, this is a curve on the unit sphere.

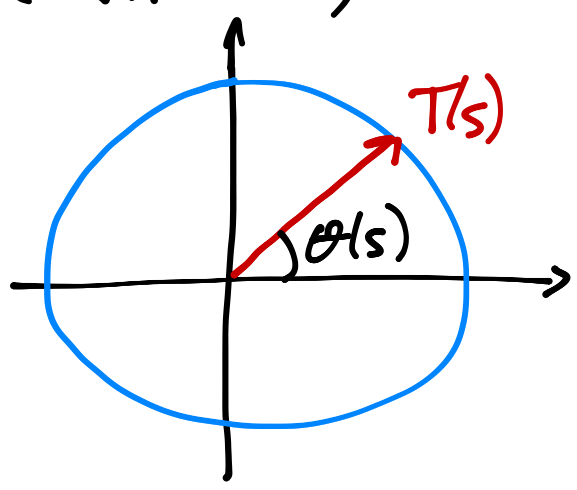


Even though $\alpha(s)$ is unit speed, $T(s)$ is probably not. In fact, from homework problem 3, $|T'(s)| = \kappa(s)$.

And the length of the tangent indicatrix is the total curvature of α:

$$\text{total curvature} = \int_0^\ell \kappa(s)\,ds = \int_0^\ell |T'(s)|\,ds = \text{length}(T(s))$$

For plane curves, $T(s)$ lies on the unit circle, so it can be described by some angle $\theta(s)$:



While θ is only well-defined up to a multiple of 2π, we can certainly require that $\theta(0)$ is b/w 0 & 2π, so then...

(i) $\theta'(s)$ is well-defined

(ii) $\theta(s) = \int_0^s \theta'(u)\,du + \theta(0)$ defines $\theta(s)$ uniquely

Df: The rotation index of a plane curve $\alpha(s)$ is given by

$$I = \frac{1}{2\pi}\left(\theta(\ell) - \theta(0)\right)$$

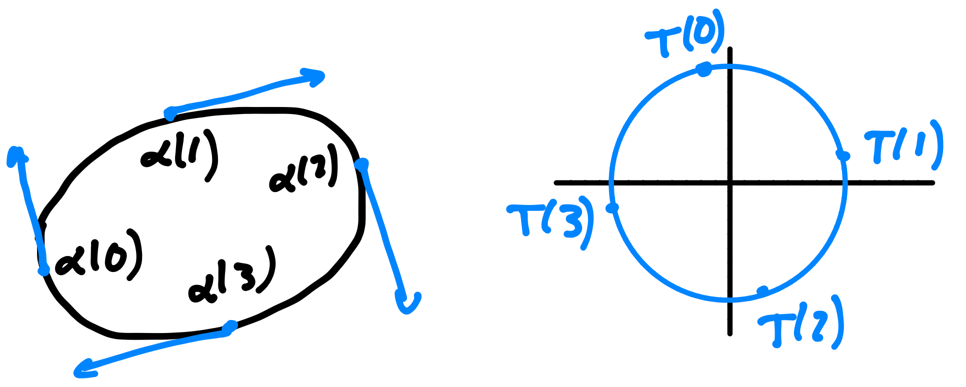
where ℓ is the length of α.

Then...

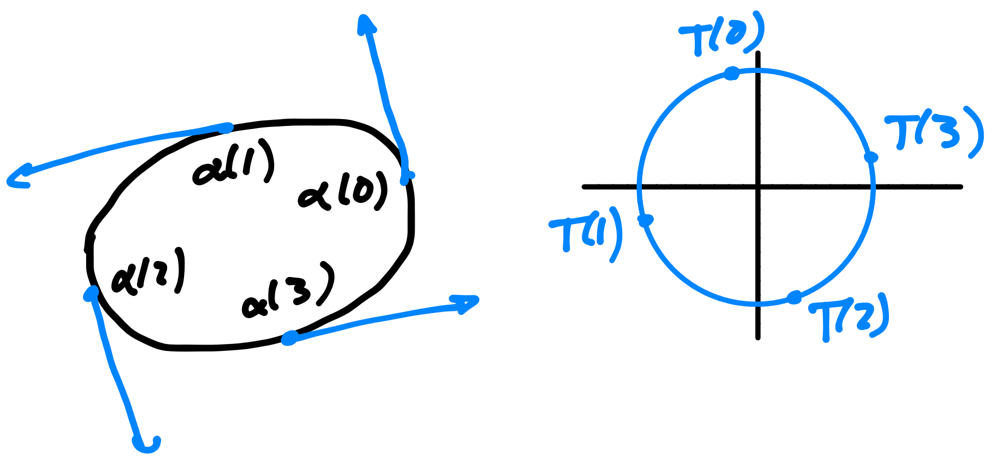
Prop: If $\alpha(s)$ is a closed plane curve, its rotation index I is an integer.

Pf: $T(s)$ must be closed, so $\theta(0)$ & $\theta(\ell)$ must differ by a mult. of 2π. ∎

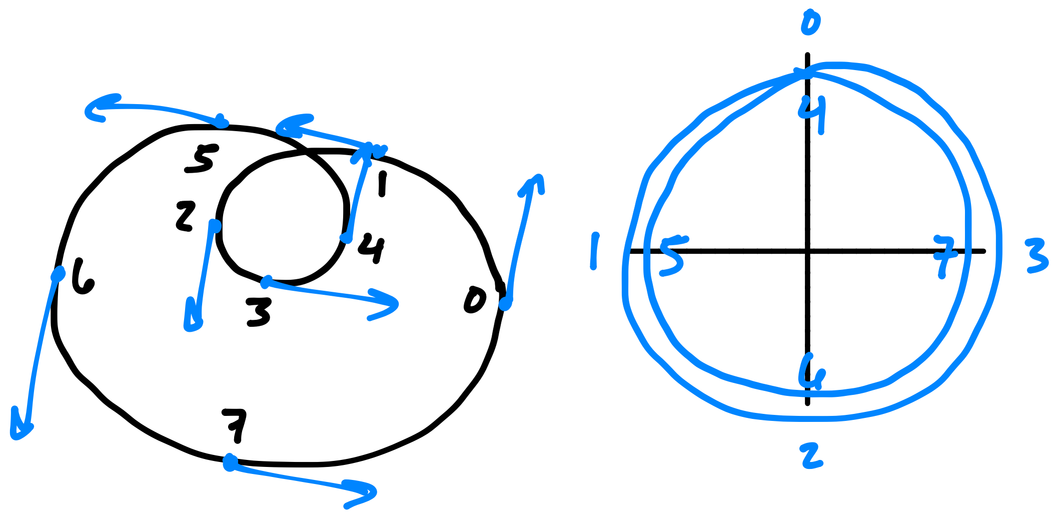
Examples:



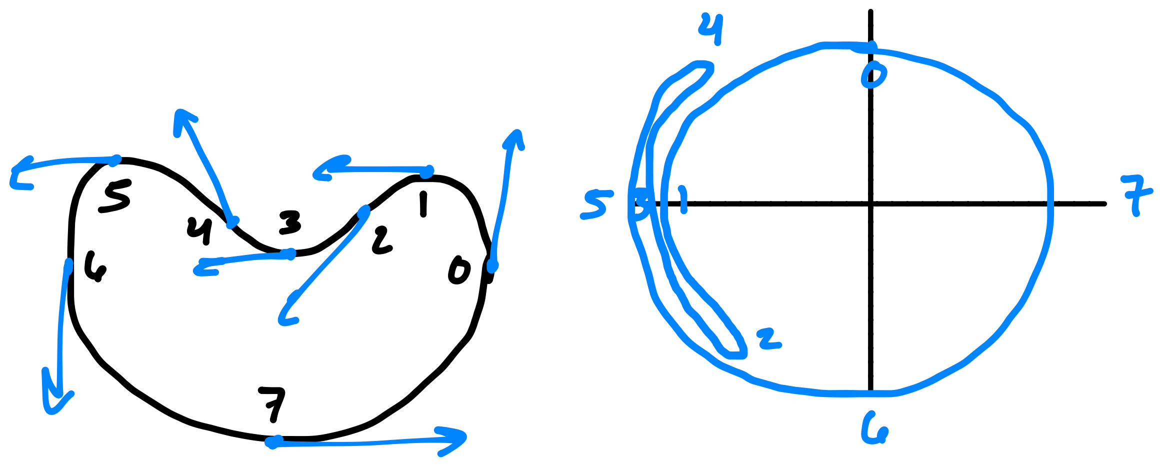
$I = -1$



$I = +1$



$I = +2$

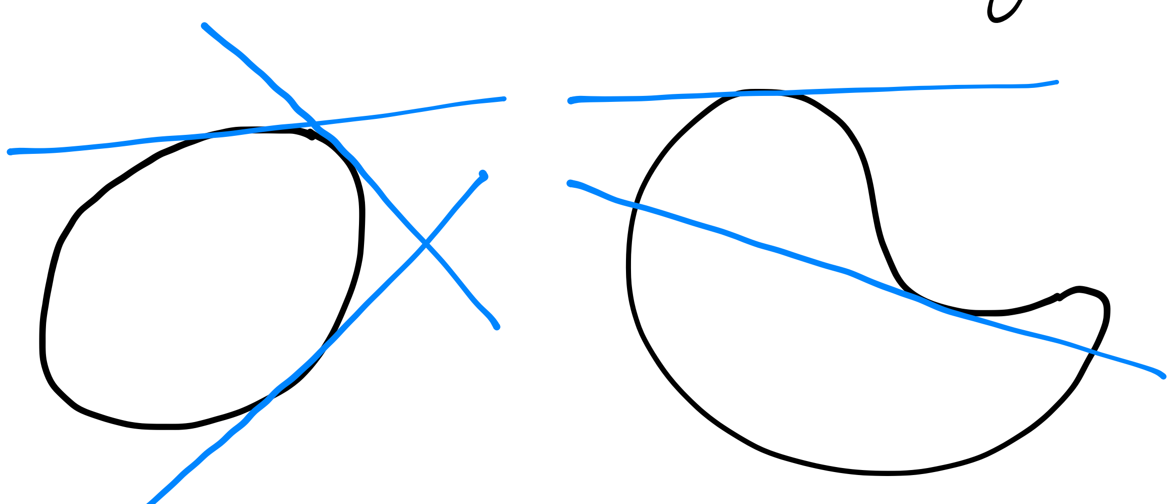


$I = +1$

Thm: A simple closed plane curve (i.e., w/ no self-intersections) has rotation index ± 1.

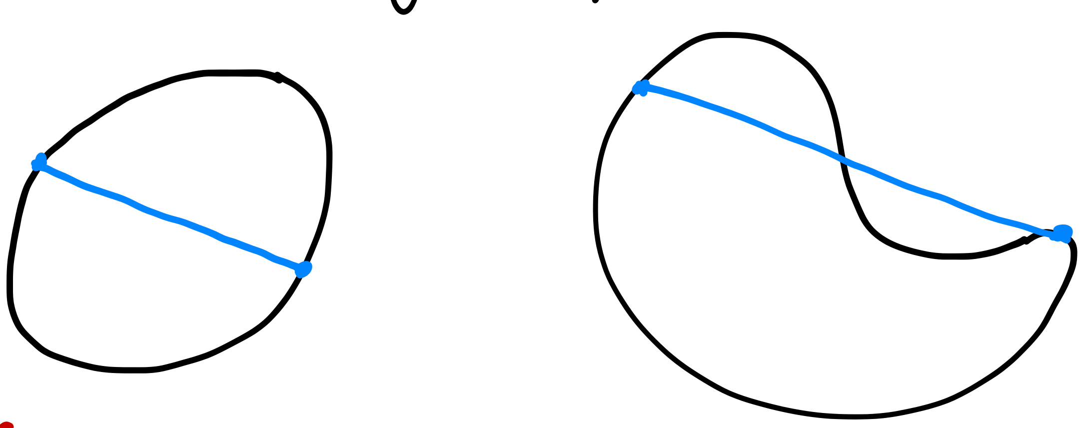
The idea is that it costs as much angle to escape from a "pocket" as you gained by going in.

Def: A plane curve is convex if it lies on one side of each of its tangent lines.



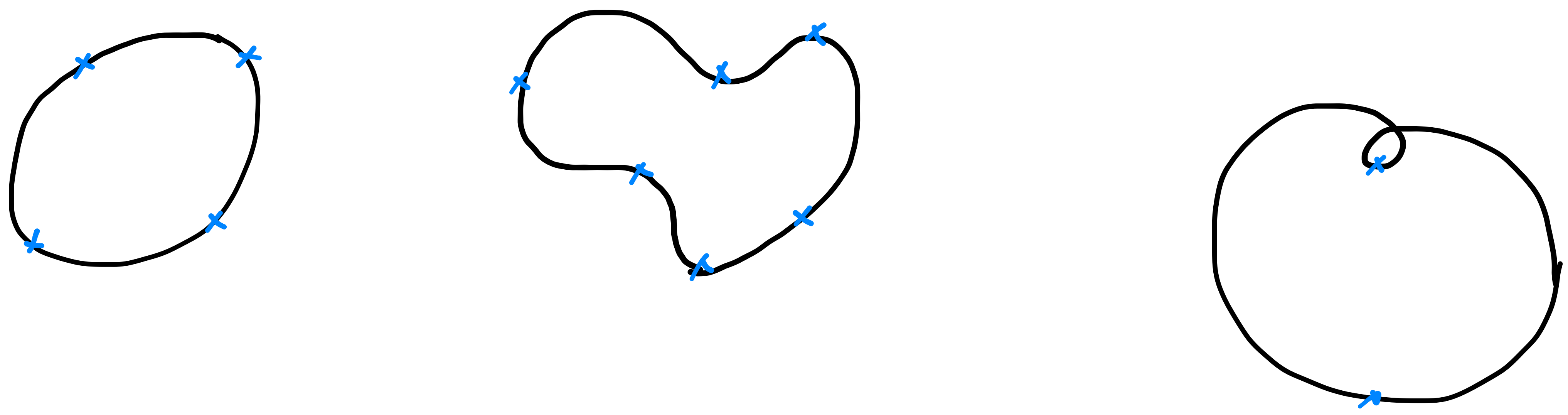
Convex Non-convex

Equivalently, a curve is convex if any two points on the curve can be connected by a line segment lying inside the curve



Convex Non-convex

Def: A **vertex** of a plane curve $\alpha(s)$ is a point where $\kappa'(s)=0$ (a critical point of curvature)

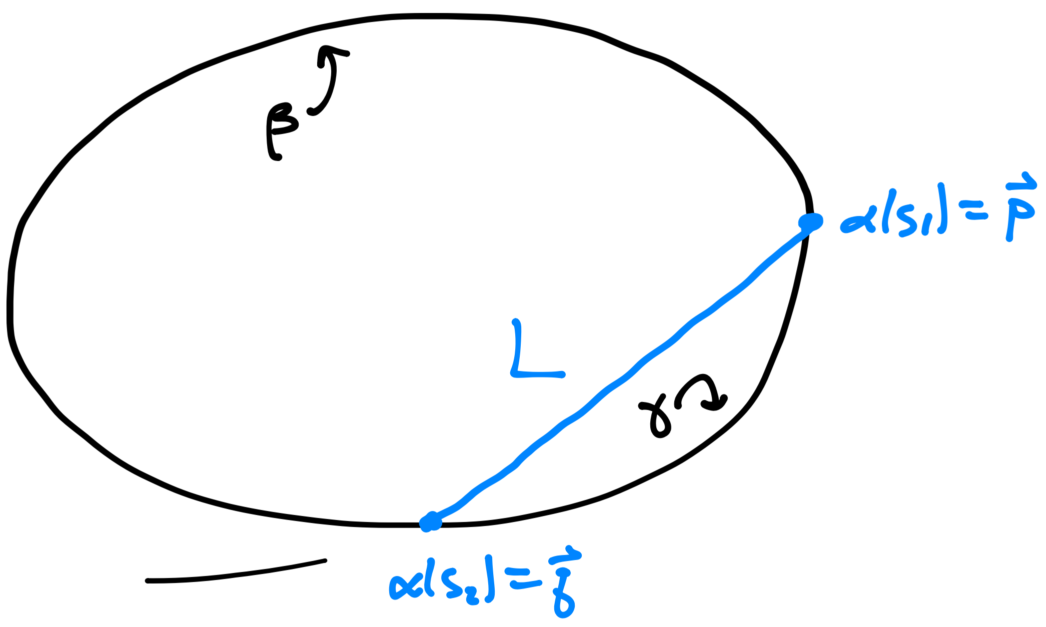


Four Vertex Thm: A simple closed plane curve has at least 4 vertices.

We will prove this in the case of convex curves, which is easier (but also maybe more surprising). Even more amazingly, the converse is true (though we won't prove it).

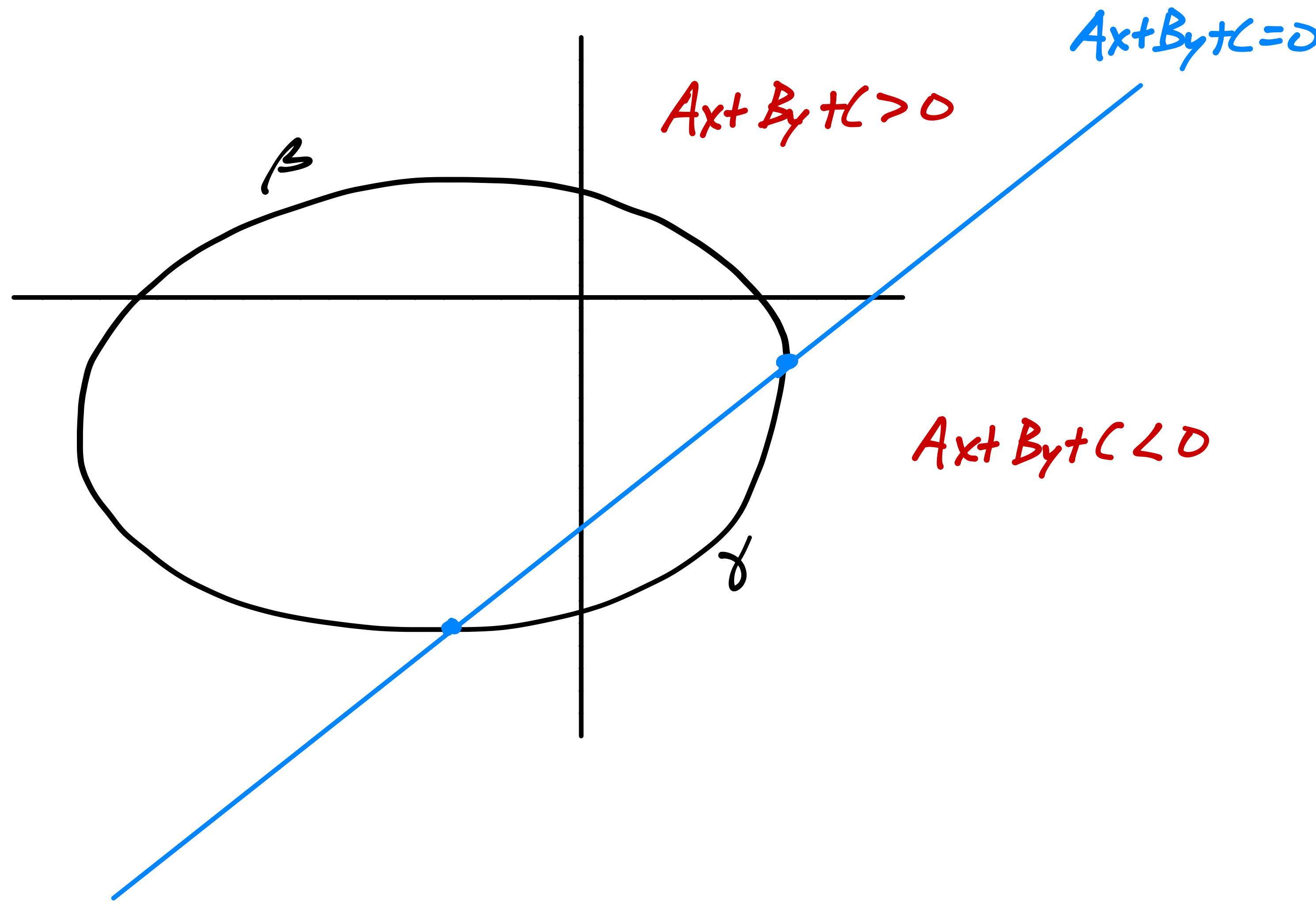
The basic idea is to prove this by contradiction. So suppose $\alpha(s)$ is parametrized by arc length on $[0,\ell]$. Since $\kappa(s)$ is continuous, it has a max & a min on $[0,\ell]$, so α has at least 2 vertices at $\alpha(s_1)=\vec{p}$ & $\alpha(s_2)=\vec{q}$.

Connect the vertices by a straight line L, & let β & γ be the two arcs of α.



Now, by convexity, β & γ lie on opposite sides of L.

At a vertex, $\kappa'(s)=0$. Now, assume for the sake of contradiction that there are no other vertices on α; then $\kappa'(s)$ must have one sign on β & the opposite sign on γ.



Now here's the tricky bit: let $Ax+By+C=0$ be the equation of L.

By switching signs of A,B,C if necessary, we have

$$(Ax(s)+By(s)+C)\,\kappa'(s)>0$$

on both β & γ. Hence,

$$\int_0^\ell (Ax(s)+By(s)+C)\,\kappa'(s)\,ds>0.$$

However, I claim this is impossible.

First of all, since α is parametrized by arclength, $\alpha'(s) = (x'(s), y'(s))$ is a unit vector, so we can write

$$\alpha'(s) = (x'(s), y'(s)) = (\cos(\theta(s)), \sin(\theta(s))).$$

and hence $\kappa(s) = |\alpha''(s)| = |(x''(s), y''(s))| = |(-\sin(\theta(s))\theta'(s), \cos(\theta(s))\theta'(s))| = \sqrt{\sin^2(\theta(s))(\theta'(s))^2 + \cos^2(\theta(s))(\theta'(s))^2} = \sqrt{(\theta'(s))^2} = \theta'(s)$,

so $\kappa'(s) = \theta''(s)$. Hence, our desired contradiction will come from:

Lemma: For any real numbers A, B, C,

$$\int_0^l (Ax(s) + By(s) + C)\,\theta''(s)\,ds = 0$$

Proof: Since $(x'(s), y'(s)) = (\cos(\theta(s)), \sin(\theta(s)))$, we have

$$x''(s) = -\sin(\theta(s))\,\theta'(s) = -y'(s)\,\theta'(s) \quad \& \quad y''(s) = \cos(\theta(s))\,\theta'(s) = x'(s)\,\theta'(s).$$

Since both $x''(s)$ & $y''(s)$ and $x'(s)$ & $y'(s)$ are periodic,

$$\int_\alpha x''(s)\,ds = 0 = \int_\alpha y''(s)\,ds.$$

But then $0 = \int_0^l x''(s)\,ds = \int_0^l -y'(s)\,\theta'(s)\,ds$. Now integrate by parts: $u = \theta'(s)$, $dv = y'(s)\,ds$, $du = \theta''(s)\,ds$, $v = y(s)$

$$= \cancelto{0}{y(l)\,\theta'(l) - y(0)\,\theta'(0)} - \int_0^l y(s)\,\theta''(s)\,ds$$

$$= -\int_0^l y(s)\,\theta''(s)\,ds$$

So we've shown that $\int_0^l y(s)\,\theta''(s)\,ds = 0$. The same game gives

$$0 = \int_0^l y''(s)\,ds = \int_0^l x'(s)\,\theta'(s)\,ds = -\int_0^l x(s)\,\theta''(s)\,ds.$$

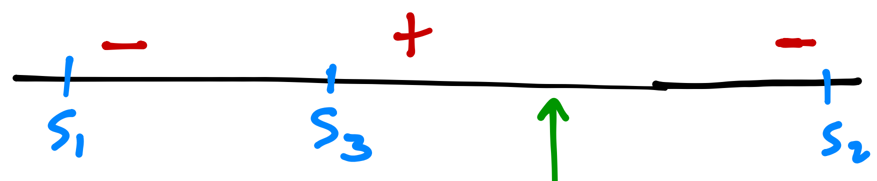
Since $\int_0^l \theta''(s)\,ds = \theta'(l) - \theta'(0) = 0$, we now have

$$\int_0^l (Ax(s) + By(s) + C)\,\theta''(s)\,ds = A\int_0^l x(s)\,\theta''(s)\,ds + B\int_0^l y(s)\,\theta''(s)\,ds + C\int_0^l \theta''(s)\,ds = 0. \qquad \blacksquare$$

In the proof of the Four Vertex Thm, we've now shown that $\kappa'(s)$ must change sign on either β or γ (or both), making a third vertex.

on β:

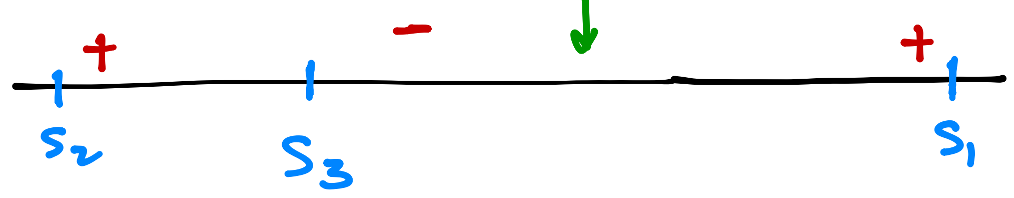
sign of $\kappa'(s)$



Since max at s_1 & min at s_2, sign of $\kappa'(s)$ is negative near both

there must be a fourth vertex in here

on γ:



Since max at s_1 & min at s_2, sign of $\kappa'(s)$ is positive near both