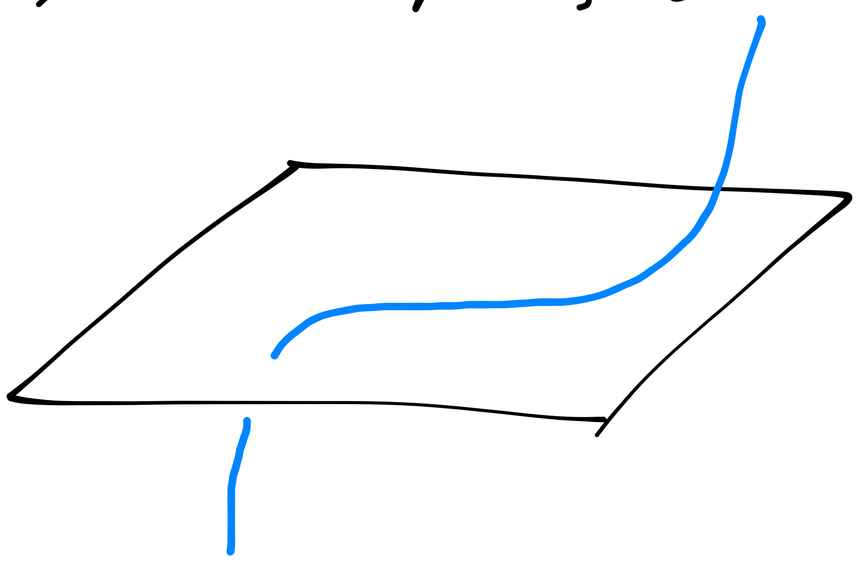
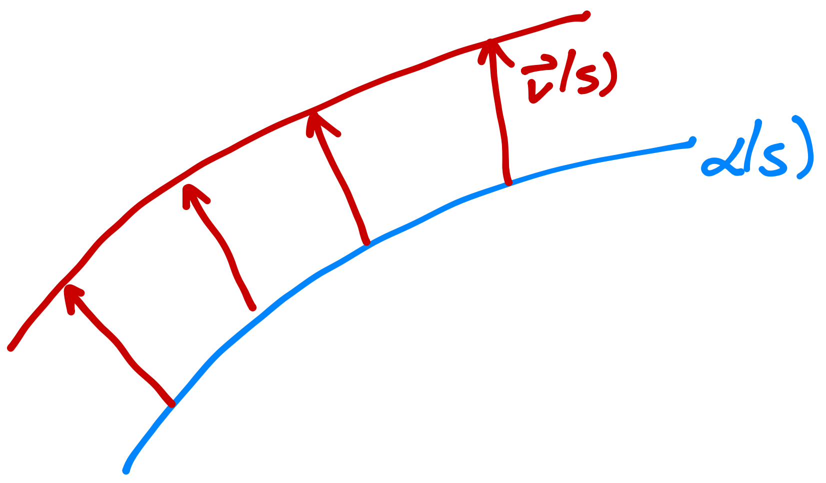
Math 474: Day 5

The Frenet frame is great, but it falls apart for curves that sometimes have zero curvature, e.g.,

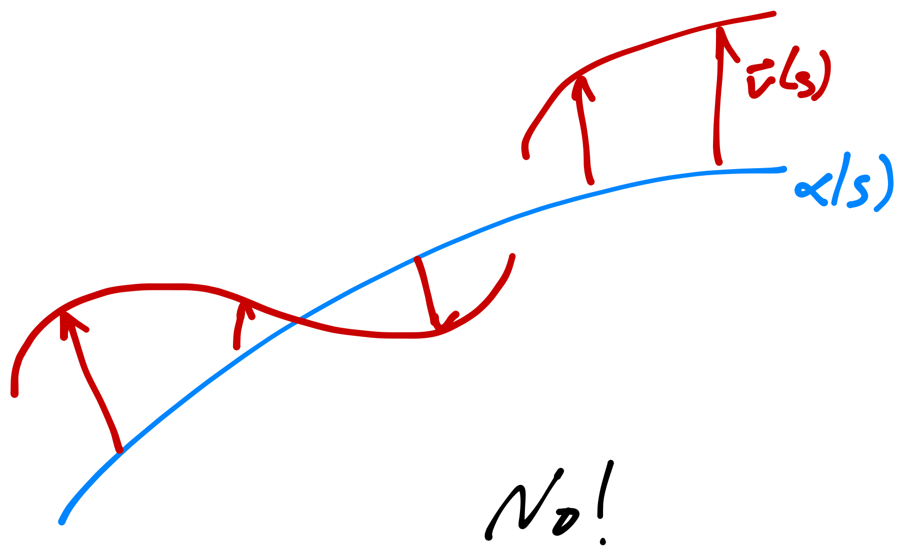


To deal w/ this, we can instead use the **Bishop frame**. Idea: given any starting frame, push it along the curve, twisting as little as possible. More precisely...

Def: A normal vector field $\vec{v}(s)$ on $\alpha(s)$ is called <u>relatively parallel</u> if its derivative is parallel to $\alpha'(s)$.



Yes!



No!

Lemma: Any relatively parallel v.f. has constant length.

Pf: It's our favorite proof in reverse: $\vec{v}(s)\cdot\alpha'(s)\equiv 0$ since $\vec{v}$ is normal to the curve. But since $\vec{v}'(s)$ is parallel to $\alpha'(s)$, this means $\vec{v}(s)\cdot\vec{v}'(s)=0$, so $0=\frac{d}{ds}(\vec{v}(s)\cdot\vec{v}(s))=\frac{d}{ds}|\vec{v}(s)|^2$. ∎

Prop: If $\alpha: I\to\mathbb{R}^3$ is a smooth curve & $s_0\in I$, then any choice of $\vec{v}(s_0)$ generates a unique relatively parallel v.f. $\vec{v}(s)$.

Proof: Uniqueness: Suppose $\vec{v}(s)$, $\vec{w}(s)$ are relatively parallel v.f.'s w/ $\vec{v}(s_0)=\vec{w}(s_0)$.

Then $\vec{v}(s)-\vec{w}(s)$ is also relatively parallel & hence has constant length. But since $\vec{v}(s_0)-\vec{w}(s_0)=\vec{0}$, this length is 0,

so $\vec{v}(s)=\vec{w}(s)$ $\forall s\in I$.

Existence: Take any smooth framing $\vec{w}_1(s),\vec{w}_2(s)$ along α, meaning that $w_1\perp\alpha$, $w_2\perp\alpha$, $w_1\perp w_2$. Then

$$\vec{v}(s)=\cos(\theta(s))\,\vec{w}_1(s)+\sin(\theta(s))\,\vec{w}_2(s)$$

for some smooth function $\theta(s)$. But now

$$\vec{v}\,'(s)=-\sin(\theta(s))\,\theta'(s)\,\vec{w}_1(s)+\cos(\theta(s))\,\vec{w}_1'(s)+\cos(\theta(s))\,\theta'(s)\,\vec{w}_2(s)+\sin(\theta(s))\,\vec{w}_2'(s).$$

Now,

$$T'(s) = \qquad\qquad p_{01}(s)\vec{w}_1(s) + p_{02}(s)\vec{w}_2(s)$$
$$\vec{w}_1'(s) = -p_{01}(s)T(s) \qquad\qquad + p_{12}(s)\vec{w}_2(s)$$
$$\vec{w}_2'(s) = -p_{02}(s)T(s) - p_{12}(s)\vec{w}_1(s)$$

For some fctns p_{01}, p_{02}, p_{12}. But then

$$\vec{v}'(s) = \left(-\sin(\theta(s))\,\theta'(s) - \sin(\theta(s))p_{12}(s)\right)\vec{w}_1(s) + \left(\cos(\theta(s))\,\theta'(s) + \cos(\theta(s))\,p_{12}(s)\right)\vec{w}_2(s) + (\text{something})\,T(s)$$

$$= \left(p_{12}(s) + \theta'(s)\right)\left(-\sin(\theta(s))\vec{w}_1(s) + \cos(\theta(s))\vec{w}_2(s)\right) + (\text{something})\,T(s)$$

But then this gives a very simple ODE for the function θ:

$$\theta'(s) = -p_{12}(s).$$

But now it's geometrically obvious that $p_{12}(s) = \vec{w}_1'(s)\cdot w_2(s)$.

Then, by the usual existence theorem for solutions of ODEs, there exists a solution $\theta(s)$, & hence the v.f. $\vec{v}$ exists. ■

Another interpretation...

Def: The **twist** of a normal v.f. $\vec{v}(s)$ is

$$Tw(\vec{v}(s), \alpha(s)) = \frac{1}{2\pi}\int \vec{v}'(s)\cdot(\alpha'(s)\times\vec{v}(s))\,ds$$

The **twist rate** is the integrand in the above.

Ex: N or B from the Frenet frame has twist rate $= \tau(s)$.

An arbitrary frame (as above) has twist rate $= p_{12}(s)$

A **relatively parallel frame** (or **Bishop frame**) has twist rate $= 0$.

Def: Given a relatively parallel unit v.f. $\vec{N}_1(s)$ along $\alpha(s)$, the associated **Bishop frame** is $(T(s) = \alpha'(s),\ N_1(s),\ N_2(s) = T(s)\times N_1(s))$.

Given such a frame, we have the corresponding Frenet equations

$$T'(s) = \qquad k_1(s)N_1(s) + k_2(s)N_2(s)$$
$$N_1'(s) = -k_1(s)T(s)$$
$$N_2(s) = -k_2(s)T(s)$$

k_1 & k_2 are sort of like the curvature & torsion. In fact, $|T'(s)| = \kappa(s) = \sqrt{k_1^2+k_2^2}$, so curvature is a sort of radius in the k_1k_2-plane.

As for torsion, note that

$$N(s) = \frac{T'(s)}{|T'(s)|} = \left(\frac{k_1}{\kappa}\right) N_1 + \left(\frac{k_2}{\kappa}\right) N_2$$

so if we write $\frac{k_1}{\kappa} = \cos\theta$ & $\frac{k_2}{\kappa} \sin\theta$, then θ' is the twist rate of $N(s)$ in the Frenet frame ... but we just said the twist rate in the Frenet frame is torsion, so

$$\theta(s) = \int_{s_0}^{s} \tau(u)\,du$$

The upshot is that κ & $\int \tau$ form a sort of polar coords. in the k_1k_2-plane.

Def: We call the curve (k_1, k_2) the *normal development* of the curve α.

Thm: If α & $\bar{\alpha}$ are regular smooth curves in $\mathbb{R}^3$ with Bishop frames having coefficients k_1, k_2 & $\bar{k}_1, \bar{k}_2$, then α is congruent to $\bar{\alpha}$ if & only if the normal developments (k_1, k_2) & $(\bar{k}_1, \bar{k}_2)$ are congruent in $\mathbb{R}^2$.