Math 474: Day 43

Isometries of the Hyperbolic Plane

Some obvious isometries of $\mathbb{H}$:

① Any horizontal translation $h:(x,y)\mapsto(x+a,y)$ (or $z\mapsto z+a$ for $a\in\mathbb{R}$)

Clearly this has no impact on the first fund. form $E=\frac{1}{y^2}=G$, $F=0$ since $dh=I_2$ & the y-coord. is unchanged.

② A scaling from the origin $h:(x,y)\mapsto(ax,ay)$ (or $z\mapsto az$)

Again, no change in 1st fund. form since $dh=\begin{bmatrix} a & 0 \\ 0 & a\end{bmatrix}$ & $\langle dh\begin{bmatrix}1\\0\end{bmatrix}, dh\begin{bmatrix}1\\0\end{bmatrix}\rangle = a^2 E_{(ax,ay)} = \frac{a^2}{a^2y^2} = \frac{1}{y^2}$ etc.

Just from these 2 we see that the isometry group of $\mathbb{H}$ acts transitively.

An obvious isometry of $\mathbb{D}$:

③ Any rotation around the origin.

$dh=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$ & $\langle dh\begin{bmatrix}0\\1\end{bmatrix}, dh\begin{bmatrix}0\\1\end{bmatrix}\rangle = \langle \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}, \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\rangle = \sin^2\theta E - 2\sin\theta\cos\theta F + \cos^2\theta G$

$= (\sin^2\theta+\cos^2\theta)\frac{4}{(1-x^2-y^2)^2} = \frac{4}{(1-x^2-y^2)^2}$

In fact compositions of these 3 basic isometries give all orientation-preserving isometries of the hyperbolic plane.