Math 474: Day 42

The reason for having multiple models for the hyperbolic plane is (largely) b/c of the desire for simple geodesics.

So now consider geodesics in $\mathbb{H}$. Recall the geodesic eqns:

$$u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1 = 0$$
$$v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2 = 0$$

Remember that $E = G = 1/y^2$, $F = 0$ and that

$$\begin{pmatrix}\Gamma_{11}^1 \\ \Gamma_{11}^2\end{pmatrix} = \frac{1}{EG-F^2}\begin{bmatrix} G & -F \\ -F & E\end{bmatrix}\begin{bmatrix}\frac{1}{2}E_x \\ F_x - \frac{1}{2}E_y\end{bmatrix} = y^4\begin{bmatrix}1/y^2 & 0 \\ 0 & 1/y^2\end{bmatrix}\begin{bmatrix}0 \\ 1/y^3\end{bmatrix} = \begin{bmatrix}0 \\ 1/y\end{bmatrix}$$

$$\begin{bmatrix}\Gamma_{12}^1 \\ \Gamma_{12}^2\end{bmatrix} = \frac{1}{EG-F^2}\begin{bmatrix} G & -F \\ -F & E\end{bmatrix}\begin{pmatrix}\frac{1}{2}E_y \\ \frac{1}{2}G_x\end{pmatrix} = \begin{bmatrix}y^2 & 0 \\ 0 & y^2\end{bmatrix}\begin{bmatrix}-1/y^3 \\ 0\end{bmatrix} = \begin{bmatrix}-1/y \\ 0\end{bmatrix}$$

$$\begin{pmatrix}\Gamma_{22}^1 \\ \Gamma_{22}^2\end{pmatrix} = \frac{1}{EG-F^2}\begin{bmatrix} G & -F \\ -F & E\end{bmatrix}\begin{bmatrix}F_y - \frac{1}{2}G_x \\ \frac{1}{2}G_y\end{bmatrix} = \begin{bmatrix}y^2 & 0 \\ 0 & y^2\end{bmatrix}\begin{bmatrix}0 \\ -1/y^3\end{bmatrix} = \begin{bmatrix}0 \\ -1/y\end{bmatrix}$$

So geodesic eqns for geodesic $\alpha(t) = (x(t), y(t))$ become

$$(\ast)\qquad x'' - \frac{2x'y'}{y} = 0$$
$$y'' + \frac{(x')^2}{y} - \frac{(y')^2}{y} = 0$$

Clearly if $x(t) = C$, then (C, Ae^{Bt}) give solutions. Of course, these are just vertical lines.

If $x' \neq 0$, then we can write $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{y'}{x'}$ &

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{\frac{d}{dt}\left(\frac{y'}{x'}\right)}{\frac{dx}{dt}} = \frac{\frac{x'y''-y'x''}{(x')^2}}{x'} = \frac{x'y''-y'x''}{(x')^3} \overset{(\ast)}{=} \frac{x'\left(\frac{(y')^2}{y} - \frac{(x')^2}{y}\right) - y'\left(\frac{2x'y'}{y}\right)}{(x')^3}$$

$$= \frac{1}{y}\left(-1 - \frac{(y')^2}{(x')^2}\right) = -\frac{1}{y}\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

So we have $\frac{d^2y}{dx^2} = -\frac{1}{y} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 \iff y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -1 \iff \frac{d}{dx}\left(y\frac{dy}{dx}\right) = -1$

Since this is a separable differential eqn, we can just integrate twice to get

$$\frac{y^2}{2} + C_1x + C_2 = \frac{-x^2}{2} + D_1x + D_2$$

or, in nicer form, $x^2 + y^2 = ax + b$.

After manipulation, this is just

$$\left(x-\frac{a}{2}\right)^2+y^2=b+\frac{a^2}{4},$$

So we see that these geodesics are semicircles centered on the x-axis.

Distances b/w pts: Suppose $P=(x,a)$ & $Q=(x,b)$. Then the geodesic b/w them is given by

$$\alpha(t)=(x,\, ae^{ct})$$

for some constant c. But the geodesic has to be parametrized by arclength, so

$$1=|\alpha'(0)|^2=|(0,\, ace^{c\cdot 0})|^2=G\cdot a^2c^2=\frac{1}{a^2}\cdot a^2c^2=c^2$$

Therefore, $d(P,Q)=|t_0|$ where $\alpha(t_0)=Q=(x,b) \iff ae^{t_0}=b \iff t_0=\ln(b/a) \iff d(P,Q)=|\ln(b/a)|$.

So, for example, you see that if $b\to 0$, then $d(P,Q)\to\infty$.

Of course, we can use this to figure out the geodesics in $\mathbb{D}$, since isometries must map geodesics to geodesics.

First of all, $f(z)=\frac{z-i}{z+i}$ preserves angles, so it's conformal, so it must map (generalized) circle arcs to (generalized) circle arcs.

Since the geodesics in $\mathbb{H}$ are semicircles intersecting the x-axis perpendicularly & since f sends circle arcs to circle arcs & preserves angles & since f maps the x-axis to the unit circle (the "boundary at infinity" of $\mathbb{D}$), then the images of geodesics in $\mathbb{H}$ must be circle arcs intersecting the unit circle perpendicularly.

Of course, something kind of funny happens when the endpoints of the circle are antipodal: you get a straight line segment (which we think of as an arc of a circle of infinite radius).