Math 474: Day 41

The Hyperbolic Plane

There are various models of the hyperbolic plane. The first is the upper half-space model:

Let $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ w/ the first fundamental form I given by $E = G = 1/y^2$, $F = 0$.

Then clearly $E_x = 0$, $E_y = -\frac{2}{y^3}$, $F_x = F_y = 0$, $G_x = 0$, $G_y = -\frac{2}{y^3}$, so, using HW problem from this week,

$$K = \frac{-1}{2\sqrt{EG}}\left[\left(\frac{E_y}{\sqrt{EG}}\right)_y + \left(\frac{G_x}{\sqrt{EG}}\right)_x\right] = \frac{-1}{2\sqrt{1/y^4}}\left[\left(\frac{-2/y^3}{\sqrt{1/y^4}}\right)_y + 0\right] = \frac{-y^2}{2}\left[\left(\frac{-2}{y}\right)_y\right] = \frac{-y^2}{2}\cdot\frac{2}{y^2} = -1.$$

So the (upper half-plane model of) hyperbolic plane has constant Gaussian curvature.

Another model: the Poincaré disk model:

Let $\mathbb{D} = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 < 1\}$ w/ the first fundamental form I given by $E = G = \frac{4}{(1-u^2-v^2)^2}$, $F = 0$.

Then $E_u = \frac{16u}{(1-u^2-v^2)^3} = G_u$, $E_v = \frac{16v}{(1-u^2-v^2)^2} = G_v$, $F_u = F_v = 0$, so we get

$$K = \frac{-1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right] = \frac{-(1-u^2-v^2)^2}{8}\left[\left(\frac{(1-u^2-v^2)^2\cdot 16v}{4(1-u^2-v^2)^3}\right)_v + \left(\frac{(1-u^2-v^2)^2\cdot 16u}{4(1-u^2-v^2)^3}\right)_u\right]$$

$$= -\frac{(1-u^2-v^2)^2}{8}\left[\left(\frac{4v}{1-u^2-v^2}\right)_v + \left(\frac{4u}{1-u^2-v^2}\right)_u\right]$$

$$= -\frac{\cancel{(1-u^2-v^2)^2}}{\cancel{8}\,2}\left[\frac{\cancel{4}(1-u^2+v^2)}{\cancel{(1-u^2-v^2)^2}} + \frac{\cancel{4}(1-v^2+u^2)}{\cancel{(1-u^2-v^2)^2}}\right]$$

$$= -\frac{2}{2} = -1.$$

So again, we have constant curvature -1.

In fact, I claim these two models are isometric. It is convenient to use complex numbers, so let

$$z = x+iy \in \mathbb{H} \quad \& \quad w = u+iv \in \mathbb{D}.$$

Then define $f: \mathbb{H} \to \mathbb{D}$ by $f(z) = \frac{z-i}{z+i}$.

Claim: f is an isometry.

Proof: First of all, note that $|f(z)|^2 = \left|\frac{z-i}{z+i}\right|^2 = \left(\frac{z-i}{z+i}\right)\left(\frac{\bar z+i}{\bar z-i}\right) = \frac{z\bar z - i\bar z + iz + 1}{z\bar z - iz + i\bar z + 1} = \frac{x^2+y^2-2y+1}{x^2+y^2+2y+1} = 1 - \frac{4y}{x^2+(y+1)^2} < 1$

since we know $y>0$. Therefore, f really does map $\mathbb{H}$ to $\mathbb{D}$. Now, f is clearly smooth (in fact, complex-analytic on the domain) & has inverse $g(w) = i\frac{1+w}{1-w}$, since

$$f\circ g(w) = f\left(i\frac{1+w}{1-w}\right) = \frac{i\frac{1+w}{1-w} - i}{i\frac{1+w}{1-w}+i} = \frac{\frac{i(1+w)-i(1-w)}{\cancel{1-w}}}{\frac{i(1+w)+i(1-w)}{\cancel{1-w}}} = \frac{2iw}{2i} = w$$

&

$$g\circ f(z) = g\left(\frac{z-i}{z+i}\right) = i\,\frac{1+\frac{z-i}{z+i}}{1-\frac{z-i}{z+i}} = i\,\frac{\frac{z+\cancel{i}+z-\cancel{i}}{\cancel{z+i}}}{\frac{\cancel{z}+i-(\cancel{z}-i)}{\cancel{z+i}}} = i\cdot\frac{2z}{2i} = z.$$

Now, if we write out $f(x,y) = \left(\frac{x^2+y^2-1}{x^2+(1+y)^2}, \frac{-2x}{x^2+(1+y)^2}\right)$, then we see that

$$df = \frac{1}{(x^2+(1+y)^2)^2}\begin{bmatrix} 4x(1+y) & 2(1+y)^2-2x^2 \\ 2x^2-2(1+y)^2 & 4x(1+y)\end{bmatrix},$$ so ... (Mathematica).