Math 474: Day 4

Having defined curvature & torsion for curves w/ arbitrary parametrizations, we now want **formulas** for curvature & torsion in such parametrizations. The definition isn't very useful, b/c the whole point is that reparametrizing by arc length is hard (since solving differential equations is hard).

Here are formulas for curvature & torsion of an arbitrary parametrized curve $\alpha(t)$:

$$\kappa(t) = \frac{|\alpha'(t)\times\alpha''(t)|}{|\alpha'(t)|^3}$$

$$\tau(t) = -\frac{(\alpha'(t)\times\alpha''(t))\cdot\alpha'''(t)}{|\alpha'(t)\times\alpha''(t)|^2} = \frac{\alpha'(t)\cdot(\alpha''(t)\times\alpha'''(t))}{|\alpha'(t)\times\alpha''(t)|^2}$$

The proof of the first is in the text, but it's painful enough that I don't want to do it all in class. The basic idea is the following:

If $s(t)$ is arclength, then $\beta(s)$ is the arclength parametrization and $\alpha(t)=\beta(s(t))$. Then $\kappa=|\beta''(s)|$, so we want to solve for $\beta''(s)$ somehow. To do so, start differentiating:

$$\alpha'(t) = \frac{d}{dt}(\beta(s(t))) = \beta'(s)\cdot s'(t) = v(t)\,T(s(t))$$

$$\alpha''(t) = \ldots = v'T + \kappa v^2 N$$

where $v(t)=s'(t)$ is the speed of the original curve. Then $|\alpha'\times\alpha''| = v^3\kappa$, so $\kappa = \frac{|\alpha'\times\alpha''|}{v(t)^3} = \frac{|\alpha'\times\alpha''|}{|\alpha'(t)|^3}$.

The formula for torsion is more of the same.

Example: A curve of constant curvature & constant (nonzero) torsion is a helix

(In fact, the converse is also true)

Suppose $\kappa(s)\equiv\kappa$ & $\tau(s)\equiv\tau$. Consider $\gamma(s)=\alpha''(s)=\kappa N(s)$. Goal #1 is to prove that $\gamma(s)$ is a circle:

1. $|\gamma(s)|=\kappa$ is constant, so $\gamma(s)$ lives on a sphere of radius κ.

2. **Claim**: γ is planar.

   Pf: Compute $\gamma'(s) = \frac{d}{ds}(\kappa N(s)) = \kappa N'(s) = \kappa(-\kappa T(s) - \tau B(s)) = -\kappa^2 T(s) - \kappa\tau B(s)$

   $$\gamma''(s) = -\kappa^3 N(s) - \kappa\tau^2 N(s) = -\kappa(\kappa^2+\tau^2)N(s)$$

   $$\gamma'''(s) = \kappa^2(\kappa^2+\tau^2)T(s) + \kappa\tau(\kappa^2+\tau^2)B(s) = -(\kappa^2+\tau^2)(-\kappa^2 T(s) - \kappa\tau B(s)) = -(\kappa^2+\tau^2)\gamma'(s)$$

   But then $\tau_\gamma(s) = \frac{\gamma'(s)\cdot(\gamma''(s)\times\gamma'''(s))}{|\gamma'(s)\times\gamma''(s)|^2} = 0$, so γ has zero torsion & hence is planar.

So $\gamma(s)$ is restricted to the intersection of a plane & the sphere of radius κ, meaning it lies on a circle.

Moreover, since $|\gamma'(s)| = |-\kappa^2 T - \kappa\tau B| = \sqrt{\kappa^4 + \kappa^2\tau^2} = \kappa\sqrt{\kappa^2+\tau^2}$ is constant, we see that γ is just a constant-speed parametrization of that circle. After applying a rigid motion, we may as well assume the circle is in the xy-plane, so

$$\alpha''(s) = \gamma(s) = \left(\kappa \cos\left(\sqrt{\kappa^2+\tau^2}\, s\right), \kappa \sin\left(\sqrt{\kappa^2+\tau^2}\, s\right), 0\right)$$

Integrating twice w.r.t. s gives

$$\alpha(s) = \left(\frac{-\kappa}{\kappa^2+\tau^2}\cos\left(\sqrt{\kappa^2+\tau^2}\, s\right), \frac{-\kappa}{\kappa^2+\tau^2}\sin\left(\sqrt{\kappa^2+\tau^2}\, s\right), Cs + D\right)$$

We can choose coordinates (i.e., shift s) so that $\gamma(0) = \left(\frac{-\kappa}{\kappa^2+\tau^2}, 0, 0\right)$, so $D=0$. Finally, solving for C gives

$$1 = |\alpha'(s)|^2 = \frac{\kappa^2}{\kappa^2+\tau^2} + \frac{\kappa^2}{\kappa^2+\tau^2} + C^2, \text{ or } C^2 = \tau^2 - \kappa^2$$

& we really do have a helix.

So having constant curvature and constant torsion is very restrictive... we only get helices. But assuming $\kappa \equiv C$ or $\tau \equiv C$ turns out not to be very restrictive at all!

Thm (Ghomi, 2006) If α is a curve of maximum curvature K & $K_2 \geq K$, then, for any given $\varepsilon > 0$, there is a curve α_2 of constant curvature K_2 so that

$$|\alpha - \alpha_2| < \varepsilon \quad \& \quad |\alpha' - \alpha_2'| < \varepsilon$$

A similar statement holds for curves of constant torsion. Even more surprisingly, you can assume the approximation is closed if the initial curve was.