Math 474: Day 38

Recall from before break that we proved that the geodesic curvature $\kappa_g(s)$ of a curve $\alpha(s)$ on a surface Σ is given by

$$\kappa_g(s) = \frac{1}{2\sqrt{EG}}\left[G_u v' - E_v u'\right] + \varphi'(s),$$

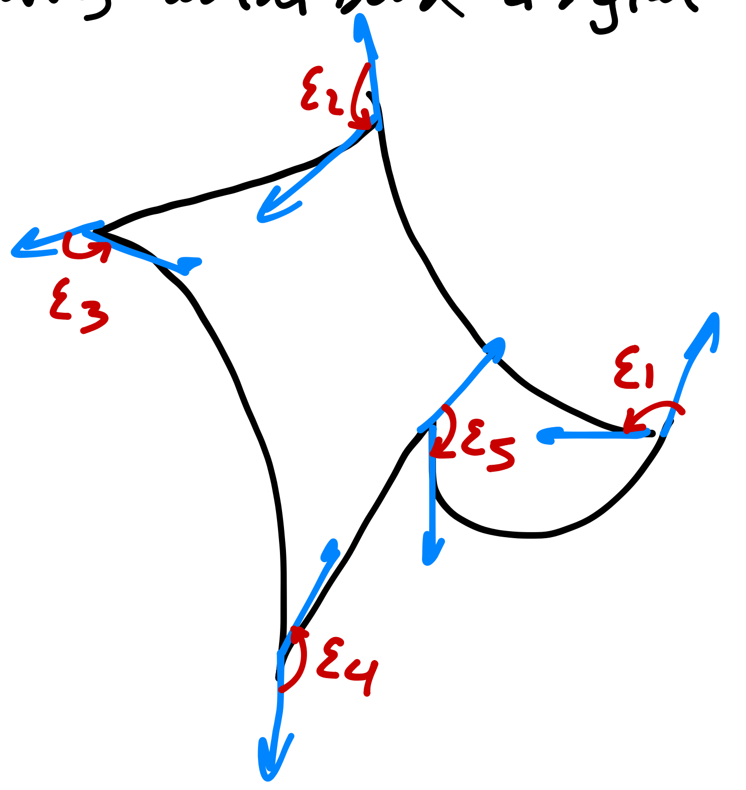
where $\varphi(s)$ is the angle b/w $\alpha'(s)$ & $\vec{x}_u$ & $\alpha(s) = \vec{x}(u(s), v(s))$.

The next HW will include the formula

$$K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

when $F=0$. But then we can combine these two slightly gross formulas to do something magical.

Suppose $\alpha_1, \ldots, \alpha_n$ are regular curves which bound a region R in Σ.



Furthermore, assume that

(i) $R = \vec{x}(P)$ for some open set P in the uv-plane which is simply connected (no holes).

(ii) We have $F=0$ on an open set containing the α_i & R.

At each corner, we can define a (signed) angle $\varepsilon_i \in (-\pi, \pi)$.

We can also define $\varphi_i(s)$ to be the angle b/w $\vec{x}_u$ & α_i' on each segment. Then

Hopf's Umlaufsatz The total turning of the tangent vector along α is 2π, which is to say

$$\sum_i \int \varphi_i'(s)\,ds + \sum_i \varepsilon_i = 2\pi.$$

(The proof is rather hard, and was only published in 1938. There's a sketch in Shifrin).

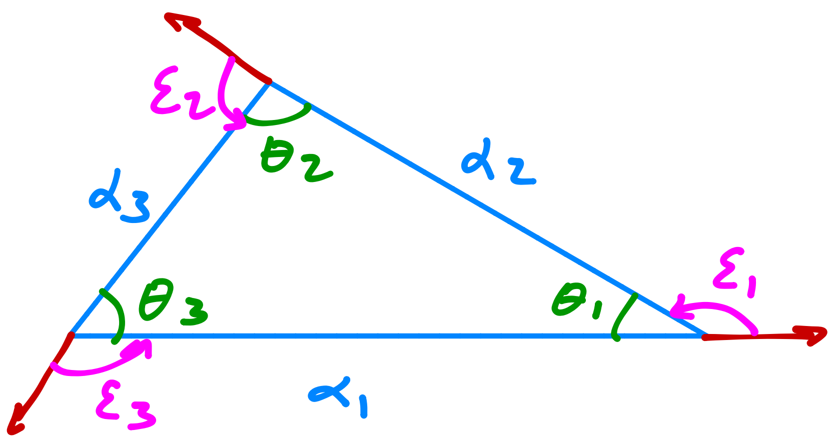
Using this, we will prove:

(Local) Gauss-Bonnet theorem:

$$\sum_i \int_{\alpha_i} \kappa_g(s)\,ds + \iint_R K\, d\text{Area} = 2\pi - \sum \varepsilon_i .$$

Now, this theorem is hopefully rather surprising, so let's look at some examples before trying to prove it.

Ex: A triangle in the plane.



$\kappa_g = 0$ on all sides & $K=0$ everywhere, so we see

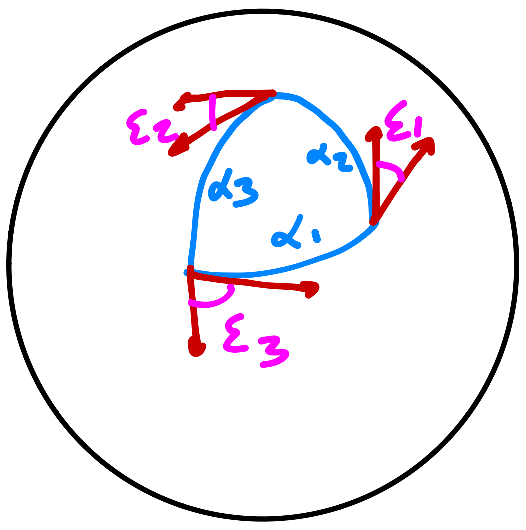
$$0 = 2\pi - \sum \varepsilon_i = 2\pi - (\pi-\theta_1) - (\pi-\theta_2) - (\pi-\theta_3) = \theta_1+\theta_2+\theta_3-\pi,$$

or $$\theta_1+\theta_2+\theta_3 = \pi$$

a fact you already know.

If you do the same w/ an n-gon, you see $\sum \theta_i = \pi(n-2)$, which again is a high school geometry fact.

Ex: A geodesic triangle on the sphere:



Again, $\kappa_g = 0$ on all sides, but now $K \equiv 1$, so we get

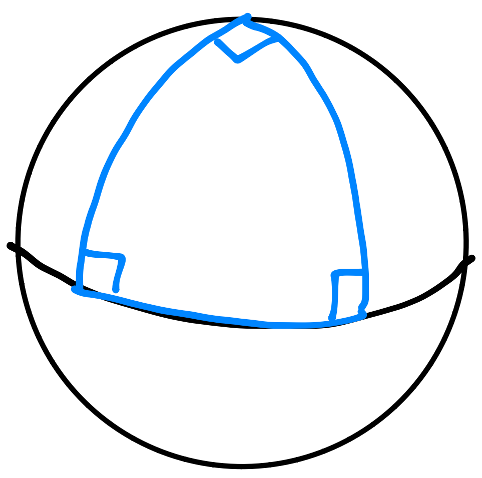
$$\iint 1\, d\text{Area} = 2\pi - \sum \varepsilon_i = \theta_1+\theta_2+\theta_3 - \pi$$

So this tells us:

① Spherical triangles have an angle sum greater than π!

② The "extra" angle is equal to the area of the triangle!

e.g.



angle sum $= \frac{3\pi}{2}$

area $= \frac{1}{8}$ of sphere $= \frac{4\pi}{8} = \frac{\pi}{2} = \frac{3\pi}{2} - \pi$.

So now we seek to prove Gauss-Bonnet. We know

$$\sum_i \int_{\alpha_i} \kappa_g(s)\,ds = \sum_i \left[\int_{\alpha_i} \frac{1}{2\sqrt{EG}}\left[G_u v' - E_v u'\right] + \int_{\alpha_i} \varphi'(s)\,ds\right]$$

From the Umlaufsatz,

$$\sum_i \int_{\alpha_i} \varphi'(s)\,ds = 2\pi - \sum \varepsilon_i$$

Now, the α_i form the boundary of R & we know that for any smooth functions P, Q on $\mathbb{R}^2$ we have

$$\int_{\partial R} P u' + Q v'\,ds = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv$$

by Green's Thm. In our case, $P = \frac{-E_v}{\sqrt{EG}}$ & $Q = \frac{G_u}{\sqrt{EG}}$ and

$$\sum \int_{\alpha_i} \frac{1}{2\sqrt{EG}}\left[G_u v' - E_v u'\right] ds = \frac{1}{2}\iint_R \left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v du\,dv$$

But then $dArea = \sqrt{EG - F^2}\,du\,dv$, so this is equal to

$$\iint_R \frac{1}{2\sqrt{EG}}\left[\left(\frac{G_u}{\sqrt{EG}}\right)_u + \left(\frac{E_v}{\sqrt{EG}}\right)_v\right] dArea = -\iint K\,dArea$$

and we conclude that

$$\sum \int_{\alpha_i} \kappa_g(s)\,ds + \iint_R K\,dArea = 2\pi - \sum \varepsilon_i$$

as desired. ▨