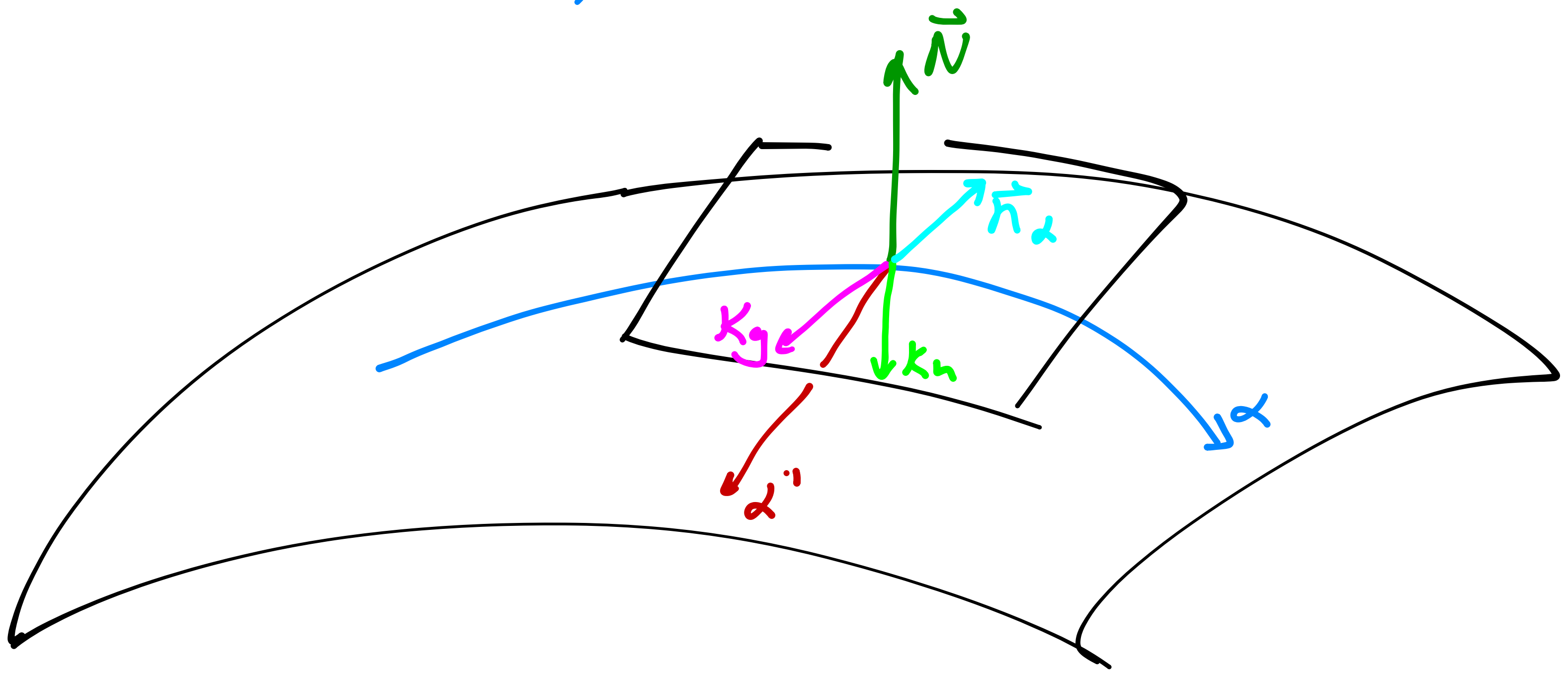
Math 474: Day 37

Recall that the projection of $\alpha''(s)$ (which is $\perp$ to $\alpha'(s)$) to the surface containing α is:

$$\alpha''(s) - \langle \vec{N}(\alpha(s)), \alpha''(s)\rangle \vec{N}(\alpha(s)) = \left(u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1\right)\vec{x}_u + \left(v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2\right)\vec{x}_v$$



Now, here's a related vector: the intrinsic normal $\vec{n}_\alpha$ of $\alpha(s)$ in Σ is a unit vector in $T_{\alpha(s)}\Sigma$ which is perpendicular to $\alpha'(s)$ & so that $\alpha'(s) \times \vec{n}_\alpha = \vec{N}(\alpha(s))$

Then, if $\alpha(s) = \vec{x}(u(s), v(s))$, the signed geodesic curvature of α is

$$\kappa_g(s) := \left\langle \begin{bmatrix} u'' \\ v'' \end{bmatrix}, \vec{n}_\alpha \right\rangle_{\mathrm{I}_{\alpha(s)}}$$

So now we prove an intrinsic formula for κ_g. Suppose $\vec{x}(u,v)$ has $F=0$ & define $\varphi(s)$ to be the angle b/w $\alpha'(s)$ & $\vec{x}_u$.

Prop: $\kappa_g(s) = \dfrac{1}{2\sqrt{EG}}\left[G_u v' - E_v u'\right] + \varphi'(s)$

Proof: Since $E = \langle \vec{x}_u, \vec{x}_u\rangle = |\vec{x}_u|^2$ & $G = \langle \vec{x}_v, \vec{x}_v\rangle = |\vec{x}_v|^2$, the vectors

$$\vec{e}_u = \frac{\vec{x}_u}{\sqrt{E}} \quad \& \quad \vec{e}_v = \frac{\vec{x}_v}{\sqrt{G}}$$

are perpendicular (since $F=0$) unit vectors, so

$$\alpha'(s) = \cos\varphi(s)\,\vec{e}_u + \sin\varphi(s)\,\vec{e}_v$$

$$\vec{n}_\alpha(s) = -\sin\varphi(s)\,\vec{e}_u + \cos\varphi(s)\,\vec{e}_v$$

But then

$$\alpha''(s) = (-\sin\varphi)\varphi'\vec{e}_u + \cos\varphi\,(\vec{e}_u)' + (\cos\varphi)\varphi'\vec{e}_v + \sin\varphi\,(\vec{e}_v)' = \varphi'\vec{n}_\alpha + \cos\varphi\,(\vec{e}_u)' + \sin\varphi\,(\vec{e}_v)'$$

Now $\kappa_g = \langle \alpha'', \vec{n}_\alpha \rangle = \varphi' + \cos\varphi \langle \vec{e}_u{}', \vec{n}_\alpha \rangle + \sin\varphi \langle \vec{e}_v{}', \vec{n}_\alpha \rangle$

$$\text{(★)} \qquad = \varphi' - \sin\varphi\cos\varphi \langle \vec{e}_u{}', \vec{e}_u \rangle + \cos^2\varphi \langle \vec{e}_u{}', \vec{e}_v \rangle - \sin^2\varphi \langle \vec{e}_v{}', \vec{e}_u \rangle + \sin\varphi\cos\varphi \langle \vec{e}_v{}', \vec{e}_v \rangle$$

Now, remember that $\vec{e}_u$ & $\vec{e}_v$ are unit vectors, so $\langle \vec{e}_u{}', \vec{e}_u \rangle = 0 = \langle \vec{e}_v{}', \vec{e}_v \rangle$.

Also, since $\langle \vec{e}_u, \vec{e}_v \rangle = 0$, we have

$$0 = \frac{d}{ds} \langle \vec{e}_u, \vec{e}_v \rangle = \langle \vec{e}_u{}', \vec{e}_v \rangle + \langle \vec{e}_u, \vec{e}_v{}' \rangle,$$

so (★) simplifies as

$$\kappa_g = \varphi' + \langle \vec{e}_u{}', \vec{e}_v \rangle$$

But now what is $\vec{e}_u{}'$?

$$\vec{e}_u{}' = \frac{d}{ds} \frac{\vec{x}_u(u(s), v(s))}{\sqrt{E(u(s), v(s))}} = \frac{1}{\sqrt{E}} \left(\vec{x}_{uu} u' + \vec{x}_{uv} v' \right) - \frac{1}{2} \frac{1}{E^{3/2}} \left(E_u u' + E_v v' \right) \vec{x}_u$$

and so

$$\langle \vec{e}_u{}', \vec{e}_v \rangle = \left\langle \vec{e}_u{}', \frac{\vec{x}_v}{\sqrt{G}} \right\rangle = \frac{u'}{\sqrt{EG}} \langle \vec{x}_{uu}, \vec{x}_v \rangle + \frac{v'}{\sqrt{EG}} \langle \vec{x}_{uv}, \vec{x}_v \rangle$$

and, since $\langle \vec{x}_{uu}, \vec{x}_v \rangle = \frac{\partial}{\partial u} \underbrace{\cancel{\langle \vec{x}_u, \vec{x}_v \rangle}}_{=F=0} - \langle \vec{x}_u, \vec{x}_{vu} \rangle = -\frac{1}{2} \frac{\partial}{\partial v} \langle \vec{x}_u, \vec{x}_u \rangle = -\frac{1}{2} E_v$

$$\langle \vec{x}_{uv}, \vec{x}_v \rangle = \frac{1}{2} \frac{\partial}{\partial u} \langle \vec{x}_v, \vec{x}_v \rangle = \frac{1}{2} G_u,$$

we see that

$$\kappa_g = \varphi' + \langle \vec{e}_u{}', \vec{e}_v \rangle = \varphi' + \frac{1}{\sqrt{EG}} \left(-\frac{1}{2} E_v u' + \frac{1}{2} G_u v' \right),$$ as desired. ∎