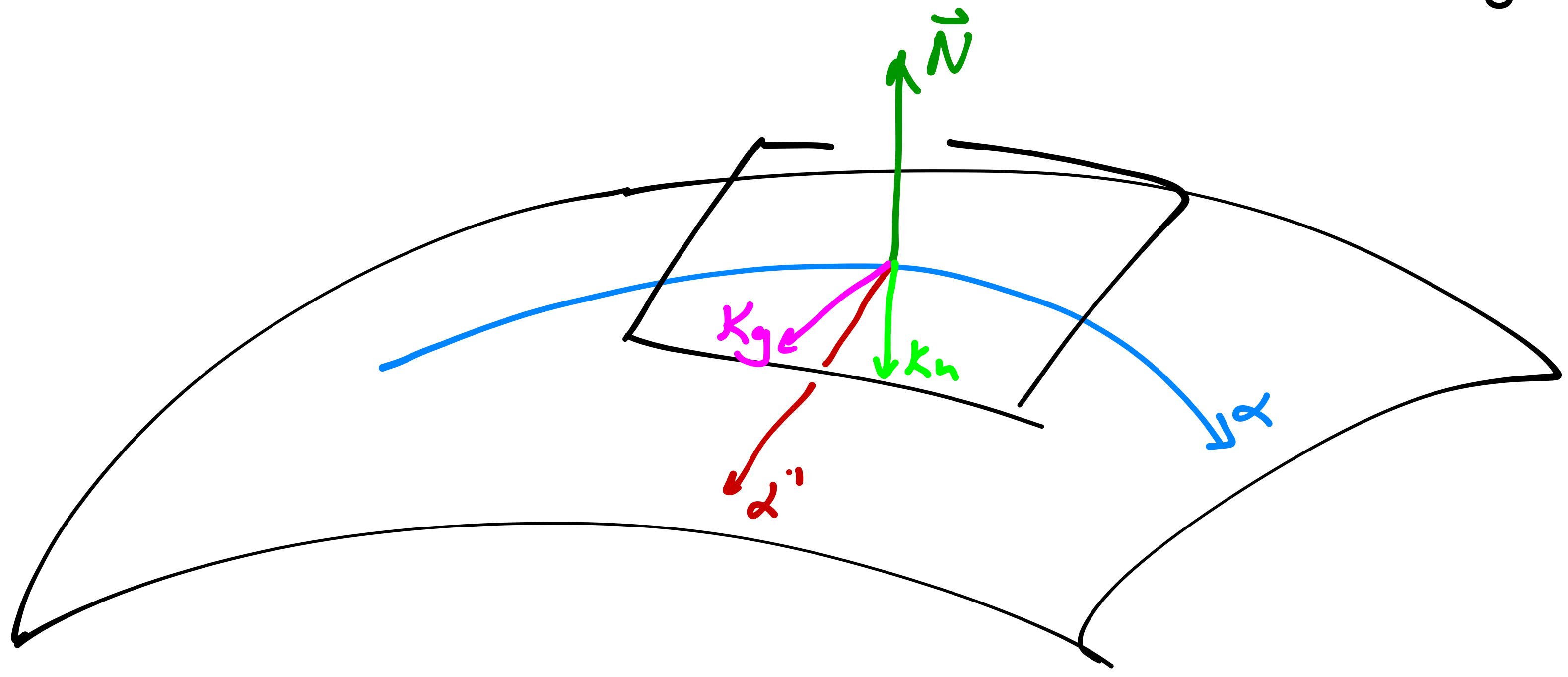
Math 474: Day 34

Geodesics

The goal now is to describe geodesics on a surface, which are the (locally) distance-minimizing curves on the surface.



Df: Given a curve $\alpha(s)$ on a surface Σ, we have

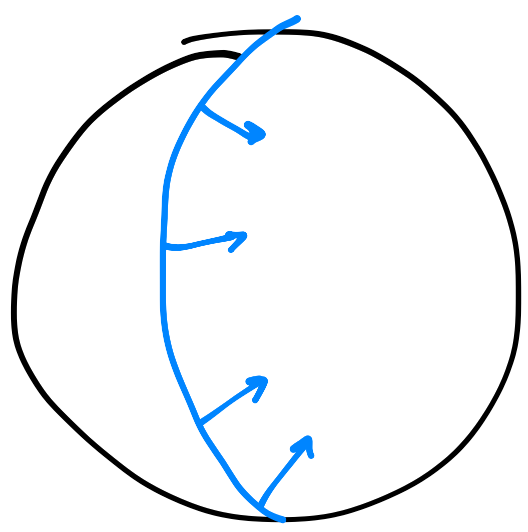
$$\alpha''(s) = \kappa_n \vec{N} + \kappa_g \vec{V}$$

where $\vec{N}$ is the surface normal at $\alpha(s)$ & $\vec{V}$ is some vector in $T_{\alpha(s)}\Sigma$.

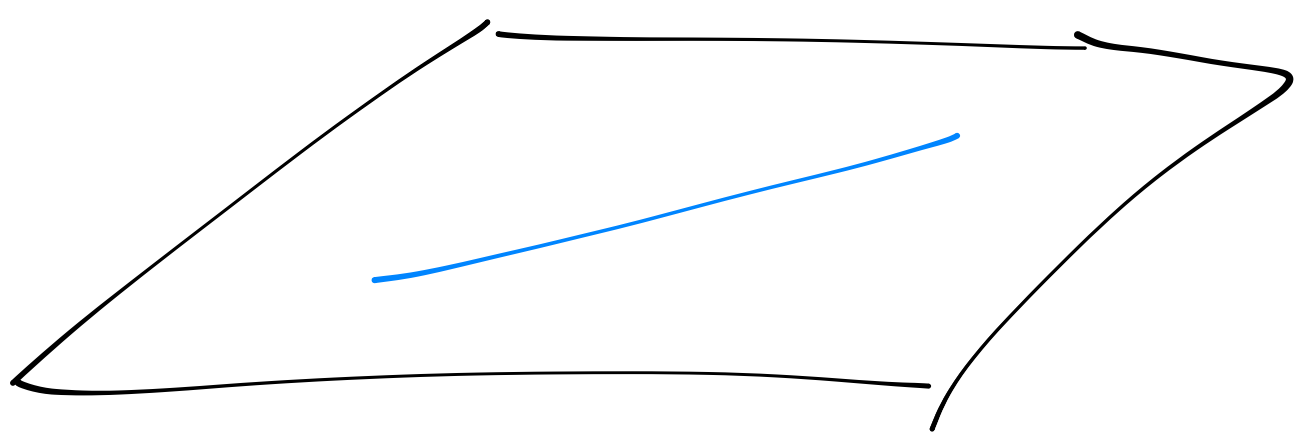
The number κ_g is the geodesic curvature of α

κ_n is the normal curvature of α.

Df: $\alpha(s)$ is a geodesic if $\kappa_g(s) \equiv 0$.

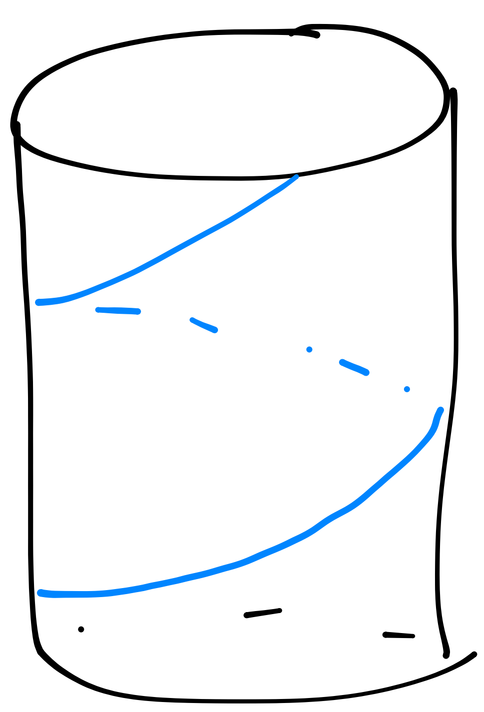


Ex: Σ the sphere, $\alpha(s)$ a great circle, since α is a normal section (& hence $\kappa \equiv \kappa_n$)



Ex: Σ a plane, $\alpha(s)$ a straight line, since $\alpha''(s) = 0$

Ex: Σ a cylinder, $\alpha(s)$ a helix.



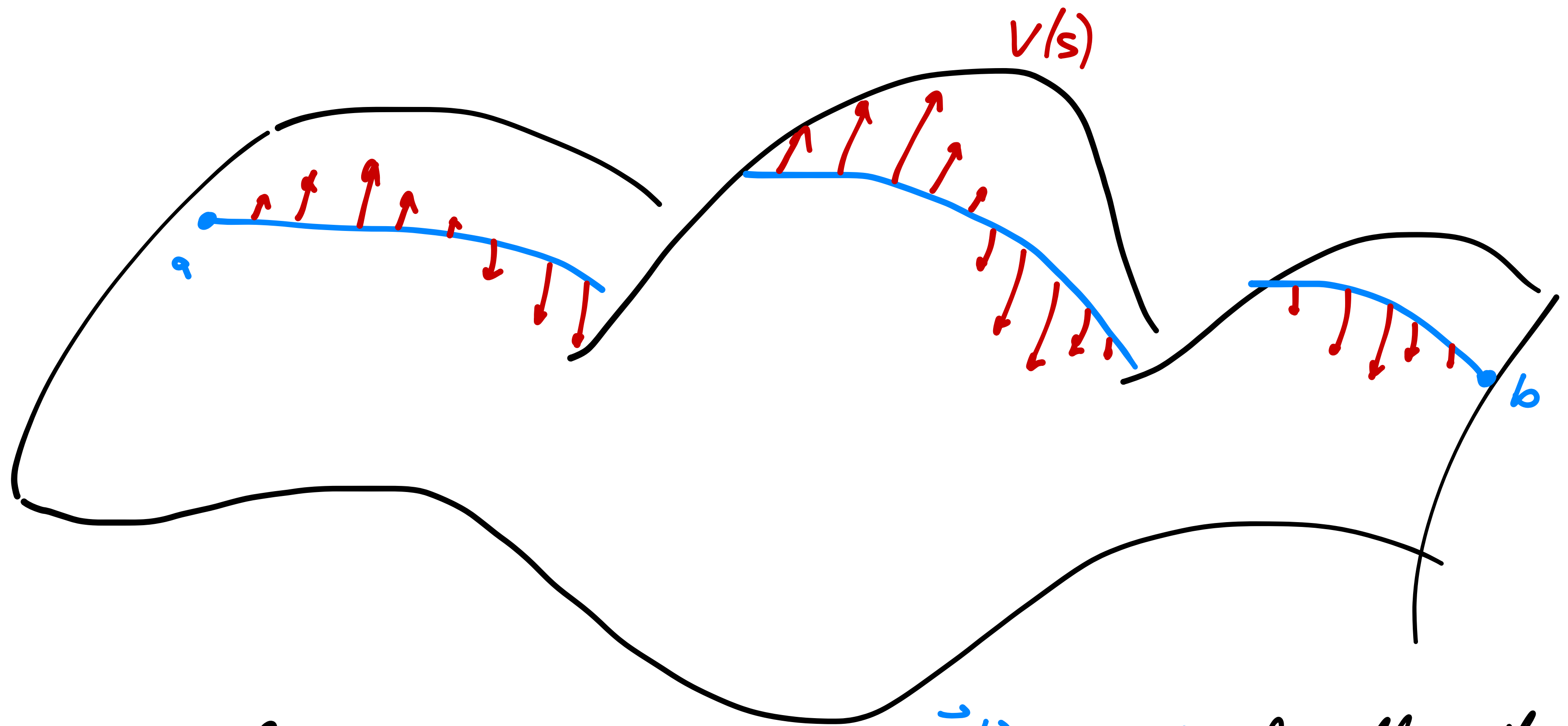
Why?

Well, if $\alpha(s) = (\cos(ks), \sin(ks), ls)$ w/ $k^2 + l^2 = 1$, then

$$\alpha'(s) = (-k\sin(ks), k\cos(ks), l)$$

$$\alpha''(s) = (-k^2\cos(ks), -k^2\sin(ks), 0),$$ which is normal to Σ.

Geodesics are the appropriate generalization of lines b/c they are locally critical for length in the following sense.



Def: If $V(s)$ is a differentiable map $\mathbb{R}\to\mathbb{R}^3$ so that $\vec V(s)\in T_{\alpha(s)}\Sigma$ for all s, then V is a vector field on $\alpha(s)$.

The variation of length of α under $\vec V$ is

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \text{length}\left(\alpha(s)+\varepsilon \vec V(s)\right) =: D_{\vec V}\,\text{len}(\alpha).$$

Prop: We have $D_{\vec v}\,\text{len}(\alpha)=0$ $\forall$ v.f. $\vec v$ on α with $\vec v(0)=\vec v(L)=\vec 0$ $\iff$ α is a geodesic joining $\alpha(0)=a$ & $\alpha(L)=b$.

Proof: First of all,

$$\text{Len}(\alpha(s)+\varepsilon\vec v(s)) = \int_0^L |\alpha'(s)+\varepsilon\vec v'(s)|\,ds,$$

so differentiating under the integral gives

$$D_{\vec v}\,\text{len}(\alpha) = \int_0^L \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} |\alpha'(s)+\varepsilon\vec v'(s)|\,ds$$

$$= \int_0^L \frac{\cancel{2}\langle \alpha'(s)+\cancel{\varepsilon\vec v'(s)}, \vec v'(s)\rangle}{\cancel{2}\|\alpha'(s)+\cancel{\varepsilon\vec v'(s)}\|}\,ds$$

$$= \int_0^L \langle \alpha'(s), \vec v'(s)\rangle\,ds$$

Now integrate by parts: since $\frac{d}{ds}\langle\alpha'(s),\vec v(s)\rangle = \langle\alpha'(s),\vec v'(s)\rangle + \langle\alpha''(s),\vec v(s)\rangle$, we know

$$D_{\vec v}\,\text{len}(\alpha) = \cancelto{0}{\langle\alpha'(s),\vec v(s)\rangle\Big|_0^L} - \int_0^L\langle\alpha''(s),\vec v(s)\rangle\,ds = -\int_0^L\langle\alpha''(s),\vec v(s)\rangle\,ds$$

($\Leftarrow$) Now, if α is a geodesic, then $\alpha''\perp T_{\alpha(s)}\Sigma$. Since $\vec v(s)\in T_{\alpha(s)}\Sigma$, we see that $\langle\alpha''(s),\vec v(s)\rangle=0$ $\forall$ s, so $D_{\vec v}\,\text{len}(\alpha)=0$.

($\Rightarrow$) OTOH, suppose $D_{\vec v}\,\text{len}(\alpha)=0$ for all $\vec v$ satisfying the boundary conditions. Then for any such $\vec v$ we have $\int_0^L\langle\alpha''(s),\vec v(s)\rangle\,ds=0$.

But this will mean $\alpha''(s)\perp T_{\alpha(s)}\Sigma$. After all, if it weren't then we could choose $\vec v(s)$ to agree w/ the projection of $\alpha''(s)$ to $T_{\alpha(s)}\Sigma$ so that $\vec v(s)=\vec 0$ outside a nbhd of $\alpha(s)$. But then $\int_0^L\langle\alpha''(s),\vec v(s)\rangle\,ds>0$, a contradiction. ▨