Math 474: Day 33

From last time we know that surfaces are (locally) isometric $\Leftrightarrow$ they have the same first fundamental form.

Now we build up to Gauss's Theorema Egregium, which says that the Gaussian curvature also only depends on the first fundamental form. This is remarkable b/c the definition of Gaussian curvature depended very much on how the surface lies in space but, as we saw w/ the catenoid & helicoid, local isometry doesn't respect position in space.

(The bad part of the Theorema Egregium is that the proof is pretty painful)

We know that $\vec{x}_u, \vec{x}_v$, & $\vec{N}$ form a basis for $\mathbb{R}^3$ at each point $p \in \Sigma$. Therefore,

$$\vec{x}_{uu} = \Gamma_{11}^1 \vec{x}_u + \Gamma_{11}^2 \vec{x}_v + L_1 \vec{N}$$

$$\vec{x}_{uv} = \Gamma_{12}^1 \vec{x}_u + \Gamma_{12}^2 \vec{x}_v + L_2 \vec{N} \qquad \text{For some } \Gamma_{ij}^k \text{ \& } L_i.$$

$$\vec{x}_{vu} = \Gamma_{21}^1 \vec{x}_u + \Gamma_{21}^2 \vec{x}_v + \overline{L}_2 \vec{N}$$

$$\vec{x}_{vv} = \Gamma_{22}^1 \vec{x}_u + \Gamma_{22}^2 \vec{x}_v + L_3 \vec{N}$$

$$\vec{N}_u = a_{11}\vec{x}_u + a_{21}\vec{x}_v$$

$$\vec{N}_v = a_{12}\vec{x}_u + a_{22}\vec{x}_v \qquad \text{(as we saw before)}$$

Moreover, $\langle \vec{x}_{uu}, N \rangle = L_1 = e$

$\langle \vec{x}_{uv}, \vec{N} \rangle = L_2 = \langle \vec{x}_{vu}, \vec{N} \rangle = \overline{L}_2 = f$

$\langle \vec{x}_{vv}, \vec{N} \rangle = L_3 = g$

Now, the Γ_{ij}^k are called Christoffel symbols, defined by

$$\frac{\partial^2}{\partial u_i \partial u_j}\vec{x} = \sum \Gamma_{ij}^k \frac{\partial}{\partial u_k}\vec{x} + (\text{stuff normal to } T_p\Sigma)$$

or, in Einstein summation notation,

$$\frac{\partial^2}{\partial u_i \partial u_j}\vec{x} = \Gamma_{ij}^k \frac{\partial x}{\partial u_k}$$

So now,

$$\frac{1}{2}E_u = \frac{1}{2}\frac{\partial}{\partial u}\langle \vec{x}_u, \vec{x}_u \rangle = \langle \vec{x}_u, \vec{x}_{uu} \rangle = \Gamma_{11}^1 \langle \vec{x}_u, \vec{x}_u \rangle + \Gamma_{11}^2 \langle \vec{x}_u, \vec{x}_v \rangle + e \langle \vec{x}_u, \vec{N} \rangle = \Gamma_{11}^1 E + \Gamma_{11}^2 F$$

Similarly, $\langle \vec{x}_{uu}, \vec{x}_v \rangle = \Gamma_{11}^1 F + \Gamma_{11}^2 G$

& $\langle \vec{x}_{uu}, \vec{x}_v \rangle = \frac{\partial}{\partial u}\langle \vec{x}_u, \vec{x}_v \rangle - \langle \vec{x}_u, \vec{x}_{vu} \rangle = F_u - \langle \vec{x}_u, \vec{x}_{uv} \rangle = F_u - \frac{1}{2}E_v$

so $\Gamma_{11}^1 F + \Gamma_{11}^2 G = F_u - \frac{1}{2}E_v$

And so on: $\langle \vec{x}_{uv}, \vec{x}_u \rangle = \Gamma_{12}^1 E + \Gamma_{12}^2 F = \frac{1}{2} E_v$

$$\langle \vec{x}_{uv}, \vec{x}_v \rangle = \Gamma_{12}^1 F + \Gamma_{12}^2 G = \tfrac{1}{2} G_u$$

$$\langle \vec{x}_{vv}, \vec{x}_u \rangle = \Gamma_{22}^1 E + \Gamma_{22}^2 F = F_v - \tfrac{1}{2} G_u$$

$$\langle \vec{x}_{vv}, \vec{x}_v \rangle = \Gamma_{22}^1 F + \Gamma_{22}^2 G = \tfrac{1}{2} G_v$$

This gives three 2x2 systems, each of which can be solved for $\Gamma_{11}^i, \Gamma_{12}^i, \Gamma_{22}^i$. For example,

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{bmatrix} \Leftrightarrow \begin{bmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{bmatrix} = \frac{1}{EG-F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{bmatrix}$$

(which is valid since $EG-F^2 \neq 0$)

This yields 6 formulas for the 6 Γ_{ij}^k; e.g.,

$$\Gamma_{11}^1 = \frac{1}{EG-F^2} \left(\tfrac{1}{2} G E_u - F F_u + \tfrac{1}{2} F E_v \right)$$

This shows that the Christoffel symbols depend only on the first fundamental form. Since isometries preserve the first fund. form, this proves

Prop: The Christoffel symbols are isometry invariants

Ex: Consider again the surface of revolution $\vec{x}(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$, which has 1st fund form

$$E = \varphi(v)^2 \qquad F = 0 \qquad G = (\varphi')^2 + (\psi')^2$$

so

$$E_u = 0 \qquad E_v = 2\varphi\varphi' \qquad F_u = F_v = 0 \qquad G_u = 0 \qquad G_v = 2\varphi'\varphi'' + 2\psi'\psi''$$

& then

$$\left.\begin{array}{l} \Gamma_{11}^1 \varphi^2 + \Gamma_{11}^2 0 = \frac{1}{2} 0 \\ \Gamma_{11}^1 0 + \Gamma_{11}^2 ((\varphi')^2 + (\psi')^2) = 0 - \varphi\varphi' \end{array}\right\} \Gamma_{11}^1 = 0, \quad \Gamma_{11}^2 = \frac{-\varphi\varphi'}{((\varphi')^2 + (\psi')^2)}, \text{ etc.}$$

Now, the more impressive fact is that some combinations of the e, f, g can be written in terms of the Γ_{ij}^k (which is surprising since the definition of e, f, g involved $\vec{N}$, but we just saw the Γ_{ij}^k depend only on (derivatives of) $\langle \vec{x}_u, \vec{x}_u \rangle, \langle \vec{x}_u, \vec{x}_v \rangle, \langle \vec{x}_v, \vec{x}_v \rangle$)

By commutativity of mixed partials,

$$(\vec{x}_{uu})_v - (\vec{x}_{uv})_u = 0 \qquad (1)$$

$$(\vec{x}_{vv})_u - (\vec{x}_{vu})_v = 0 \qquad (2)$$

$$(\vec{N}_u)_v - (\vec{N}_v)_u = 0 \qquad (3)$$

Expanding (1) in the $\vec{x}_u, \vec{x}_v, \vec{N}$ basis gives

$$A_1\vec{x}_u + B_1\vec{x}_v + C_1\vec{N} = \vec{0}$$

for some A_1, B_1, C_1. But this is a linear combination of linearly ind. vectors giving $\vec{0}$, so $A_1 = B_1 = C_1 = 0$

But now this is going to tell us something interesting!

$$(\vec{x}_{uu})_v = (\Gamma_{11}^1)_v \vec{x}_u + \Gamma_{11}^1 \vec{x}_{uv} + (\Gamma_{11}^2)_v \vec{x}_v + \Gamma_{11}^2 \vec{x}_{vv} + e_v\vec{N} + e\vec{N}_v$$

$$= (\Gamma_{11}^1)_v \vec{x}_u + \Gamma_{11}^1(\Gamma_{12}^1\vec{x}_u + \Gamma_{12}^2\vec{x}_v + f\vec{N})$$

$$+ (\Gamma_{11}^2)_v \vec{x}_v + \Gamma_{11}^2(\Gamma_{22}^1\vec{x}_u + \Gamma_{22}^2\vec{x}_v + g\vec{N})$$

$$+ e_v\vec{N} + e(a_{12}\vec{x}_u + a_{22}\vec{x}_v)$$

and

$$(\vec{x}_{uv})_u = (\Gamma_{12}^1)\vec{x}_u + \Gamma_{12}^1\vec{x}_{uu} + (\Gamma_{12}^2)_u \vec{x}_v + \Gamma_{12}^2\vec{x}_{vu} + f_u\vec{N} + f\vec{N}_u$$

$$= (\Gamma_{12}^1)_u \vec{x}_u + \Gamma_{12}^1(\Gamma_{11}^1\vec{x}_u + \Gamma_{11}^2\vec{x}_v + e\vec{N})$$

$$+ (\Gamma_{12}^2)_u \vec{x}_v + \Gamma_{12}^2(\Gamma_{21}^1\vec{x}_u + \Gamma_{21}^2\vec{x}_v + f\vec{N})$$

$$+ f_u\vec{N} + f(a_{11}\vec{x}_u + a_{21}\vec{x}_v)$$

So now, if we isolate B_1, which is the coefficient of $\vec{x}_v$ in $(\vec{x}_{uu})_v - (\vec{x}_{uv})_u = \vec{0}$, we get

$$\Gamma_{11}^1\Gamma_{12}^2 + (\Gamma_{11}^2)_v + \Gamma_{11}^2\Gamma_{22}^2 + ea_{22} = \Gamma_{12}^1\Gamma_{11}^2 + (\Gamma_{12}^2)_u + \Gamma_{12}^2\Gamma_{21}^2 + fa_{21}$$ (Gauss formula)

But then

$$ea_{22} - fa_{21} = \text{mess of Christoffel symbols}$$

By the Weingarten Equations,

$$a_{22} = \frac{fF - gE}{EG - F^2} \qquad a_{21} = \frac{eF - fE}{EG - F^2}$$

so the LHS of the Gauss eqn becomes

$$\frac{\cancel{efF} - egE - \cancel{feF} + f^2E}{EG - F^2} = -E\,\frac{eg - f^2}{EG - F^2} = -EK$$

So we've now written $-EK$ as a combination of Christoffel symbols, which we know only depend on the first fundamental form.

This proves

Theorema Egregium (Gauss): Gaussian curvature is invariant under isometries.

(Somewhat surprising) Corollary: The helicoid & catenoid have the same Gaussian curvature at corresponding points.

Repeating the above procedure for A_1 gives an expression for FK, & doing it for C_1 gives

$$\Gamma_{11}^1 f + \Gamma_{11}^2 g + e_v = \Gamma_{12}^1 e + \Gamma_{12}^2 f + f_u$$

or

$$e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2 \qquad (*)$$

Repeating for (2) gives expressions for FK & GK as well as

$$f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2 \qquad (*)$$

The equations (*) are called the Mainardi-Codazzi equations.

Recalling that the Γ_{ij}^k are expressed in terms of E, F, G, the Gauss formula plus the M-C equations are really equations relating E, F, G & e, f, g.

It turns out that these are the only such relations (this is called Bonnet's Thm).