Math 474: Day 32

Def: A smooth map $f: \Sigma_1 \to \Sigma_2$ is an **isometry** if f has a smooth inverse & if

$$\langle \vec{w}_1, \vec{w}_2 \rangle_{I_p} = \langle df_p \vec{w}_1, df_p \vec{w}_2 \rangle_{I_{f(p)}}$$

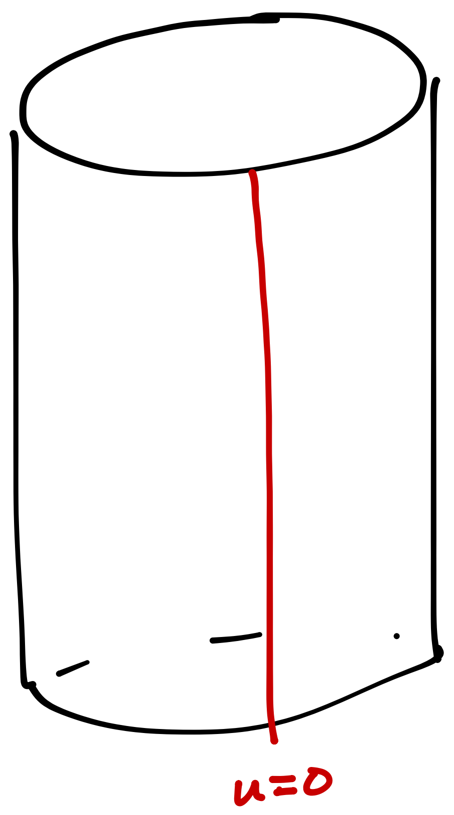
for all $p \in \Sigma_1$ & $\vec{w}_1, \vec{w}_2 \in T_p\Sigma_1$.

Equivalently, if $I_p(\vec{w}, \vec{w}) = I_{f(p)}(df_p\vec{w}, df_p\vec{w})$, then, by polarizing, we get that f is an isometry.

Def: A map $f: V \to \Sigma_2$ of a nbhd V of $p \in \Sigma_1$ is a **local isometry** at p if $\exists$ nbhd W of $f(p)$ s.t. $f: V \to W$ is an isometry.

If there exists a local isometry into Σ_2 at every $p \in \Sigma_1$, then Σ_1 & Σ_2 are **locally isometric**.

Ex: Consider the parametrization $\vec{x}(u,v) = (\cos u, \sin u, v)$ of the cylinder, where $0 < u < 2\pi$, $-\infty < v < \infty$ (Notice this really only parametrizes a subset of the cylinder)



Also, let $\vec{y}(u,v) = (u,v,0)$ be the trivial parametrization of the plane

Claim: $\vec{y} \circ \vec{x}^{-1}$ is a local isometry on the indicated open subset of the cylinder.

Proof: $\vec{x}^{-1}$ & $\vec{y}$ are clearly smooth, & bijective, so their composition is as well.

Now, let $\vec{w} = a\vec{x}_u + b\vec{x}_v \in T_pC$. Then

$$I_p(\vec{w}, \vec{w}) = a^2E + 2abF + b^2G = a^2 + b^2$$

Since $\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on the cylinder.

OTOH, if $f = \vec{y} \circ \vec{x}^{-1}$, then $f(\vec{x}(u,v)) = \vec{y}(u,v)$, so if $\vec{w} = \alpha'(0)$ for some curve $\alpha(t) = \vec{x}(u(t), v(t))$, then

(i) $u'(0) = a$, $v'(0) = b$

(ii) $df_p\vec{w}$ is tangent to $\beta(t) = f(\vec{x}(u(t), v(t))) = \vec{y}(u(t), v(t)) \iff df_p\vec{w} = \beta'(0) = (u'(0), v'(0), 0) = (a, b, 0)$,

so then $I_{f(p)}(df_p\vec{w}, df_p\vec{w}) = a^2\bar{E} + 2ab\bar{F} + b^2\bar{G} = a^2 + b^2$ since $\begin{bmatrix} \bar{E} & \bar{F} \\ \bar{F} & \bar{G} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on the plane.

This then means the cylinder & the plane are locally isometric.

It turns out they cannot be globally isometric b/c there isn't even a bijective continuous map w/ continuous inverse b/w them, let alone a smooth one that preserves the inner product.

This example suggests the following proposition

Prop: Suppose $\exists$ parametrizations $\vec{x}: U \to \Sigma_1$ & $\vec{y}: V \to \Sigma_2$ s.t. $E_1 = E_2$, $F_1 = F_2$, $G_1 = G_2$. Then the map

$f = \vec{y} \circ \vec{x}^{-1} : \vec{x}(U) \to \Sigma_2$ is a local isometry.

Proof: Let $p \in \vec{x}(U) \subseteq \Sigma_1$ & let $\vec{w} \in T_p\Sigma_1$. Then $\vec{w} = \alpha'(0)$ for some $\alpha(t) = \vec{x}(u(t), v(t))$, so

$$\vec{w} = \frac{d}{dt}\Big|_{t=0} \vec{x}(u(t), v(t)) = u'(0)\vec{x}_u + v'(0)\vec{x}_v.$$

By definition then, $df_p\vec{w} = \frac{d}{dt}\Big|_{t=0} \vec{y} \circ \vec{x}^{-1}(\alpha(t)) = \frac{d}{dt}\Big|_{t=0} \vec{y} \circ \vec{x}^{-1}\left(\vec{x}(u(t), v(t))\right) = u'(0)\vec{y}_u + v'(0)\vec{y}_v$.

So then $I_p(\vec{w}, \vec{w}) = u'(0)^2 E_1 + 2u'(0)v'(0) F_1 + v'(0)^2 G_1 = u'(0)^2 E_2 + 2u'(0)v'(0) F_2 + v'(0)^2 G_2 = I_{f(p)}(df_p\vec{w}, df_p\vec{w})$,

& hence f is a local isometry. $\blacksquare$

This is kind of amazing: it says that you can conclude that 2 surfaces have the same local geometry just by computing their first fundamental form.

Ex: Consider the catenoid, which is the surface of revolution generated by the catenary $(\varphi(v), \psi(v)) = (a\cosh v, av)$.

Then, as we saw last week, the catenoid has the parametrization $\vec{x}(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$

$= (a\cosh v\cos u, a\cosh v\sin u, av)$

& the first fundamental form coefficients are

$$E = \varphi(v)^2 = a^2\cosh^2 v$$

$$F = 0$$

$$G = (\varphi')^2 + (\psi')^2 = a^2(\sinh^2 v + 1) = a^2\cosh^2 v.$$

Now, I claim the catenoid is locally isometric to the helicoid

$$\vec{y}(u,v) = (\bar{v}\cos\bar{u}, \bar{v}\sin\bar{u}, a\bar{u})$$

To see this, make the change of coordinates $\bar{u} = u$, $\bar{v} = a\sinh v$. This is valid since the Jacobian

$\begin{bmatrix} 1 & 0 \\ 0 & a\cosh v \end{bmatrix}$ is nonsingular.

But then $\vec{y}(u,v) = (a \sinh v \cos u, a \sinh v \sin u, au)$ has first fd. form

$$E = a^2 \cosh^2 v$$

$$F = 0$$

$$G = a^2 \cosh^2 v$$

the same as the catenoid, so the prop. implies they are locally isometric