Math 474: Day 28

From last time we have the Weingarten equations in matrix form

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \frac{-1}{EG-F^2}\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

or, as a system of scalar equations:

$$a_{11} = \frac{fF-eG}{EG-F^2} \qquad a_{12} = \frac{gF-fG}{EG-F^2} \qquad a_{21} = \frac{eF-fE}{EG-F^2} \qquad a_{22} = \frac{fF-gE}{EG-F^2}$$

This gives 2 different matrix reps of II_p:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

hard to compute

dN_p as a linear map in the $\vec{x}_u, \vec{x}_v$ basis. Represents II_p w.r.t. I_p inner product

easy to compute

II_p as a quadratic form w.r.t. std dot product on $\mathbb{R}^2$

Of course, by definition $K = \det dN_p = a_{11}a_{22} - a_{21}a_{12}$

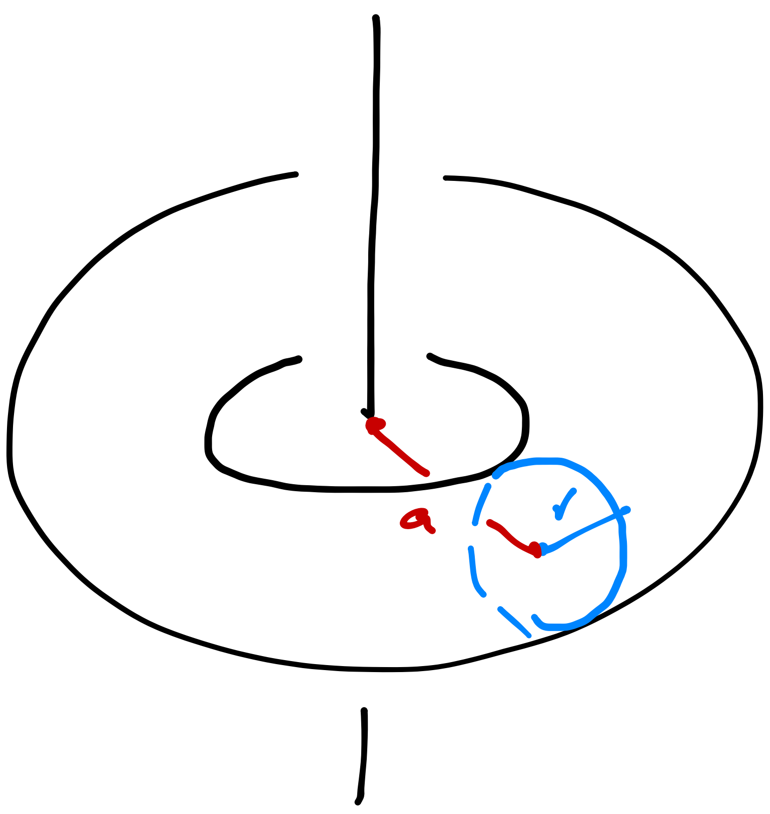
$$H = -\frac{1}{2}\operatorname{tr} dN_p = \frac{1}{2}(a_{11}+a_{22})$$

But this is much easier using the above matrix eqn:

$$K = \det\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \det\begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix} \cdot \frac{1}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} = \frac{eg-f^2}{EG-F^2}$$

$$H = -\frac{1}{2}\operatorname{tr}\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \frac{1}{2}\,\frac{eG-2fF+gE}{EG-F^2}$$ using the Weingarten equations.

Ex:



$$\vec{x}(u,v) = ((a+r\cos u)\cos v, (a+r\cos u)\sin v, r\sin u)$$

The goal is to compute K & H using E, F, G, e, f, g.

Trick: $|\vec{x}_u \times \vec{x}_v|^2 + \langle \vec{x}_u, \vec{x}_v\rangle^2 = \|\vec{x}_u\|^2\|\vec{x}_v\|^2 \Rightarrow |\vec{x}_u \times \vec{x}_v|^2 = E^2G^2 - F^2$

Now, $\vec{x}_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$

$\vec{x}_v = (-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0)$

$\vec{x}_{uu} = (-r\cos u\cos v, -r\cos u\sin v, -r\sin u)$

$\vec{x}_{uv} = (r\sin u\sin v, -r\sin u\cos v, 0)$

$\vec{x}_{vv} = (-(a+r\cos u)\cos v, -(a+r\cos u)\sin v, 0)$

$\Rightarrow$

$E = \langle \vec{x}_u, \vec{x}_u\rangle = r^2\sin^2 u\cos^2 v + r^2\sin^2 u\sin^2 v + r^2\cos^2 u = r^2$

$F = \langle \vec{x}_u, \vec{x}_v\rangle = r(a+r\cos u)\sin v\cos v, -r(a+r\cos u)\sin v\cos v = 0$

$G = \langle \vec{x}_v, \vec{x}_v\rangle = (a+r\cos u)^2\sin^2 v + (a+r\cos u)^2\cos^2 v = (a+r\cos u)^2$

Now,

$$e = \langle \vec{N}, \vec{x}_{uu} \rangle = \left\langle \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}, \vec{x}_{uu} \right\rangle = \frac{1}{\sqrt{EG-F^2}} \langle \vec{x}_u \times \vec{x}_v, \vec{x}_{uu} \rangle$$

$$f = \frac{1}{\sqrt{EG-F^2}} \langle \vec{x}_u \times \vec{x}_v, \vec{x}_{uv} \rangle$$

$$g = \frac{1}{\sqrt{EG-F^2}} \langle \vec{x}_u \times \vec{x}_v, \vec{x}_{vv} \rangle$$

So then

$$\vec{x}_u \times \vec{x}_v = \left(-r(a+r\cos u)\cos u \cos v, -r(a+r\cos u)\cos u \sin v, -r(a+r\cos u)\sin u \cos^2 v - r(a+r\cos u)\sin u \sin^2 v\right)$$

$$= -r(a+r\cos u)\left(\cos u \cos v, \cos u \sin v, \sin u\right)$$

And thus:

$$\langle \vec{x}_u \times \vec{x}_v, \vec{x}_{uu} \rangle = r^2(a+r\cos u)\left(\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u\right) = r^2(a+r\cos u)$$

$$\langle \vec{x}_u \times \vec{x}_v, \vec{x}_{uv} \rangle = -r^2(a+r\cos u)\left(\sin u \cos u \sin v \cos v - \sin u \cos u \sin v \cos v\right) = 0$$

$$\langle \vec{x}_u \times \vec{x}_v, \vec{x}_{vv} \rangle = r(a+r\cos u)^2\left(\cos u \cos^2 v + \cos u \sin^2 v\right) = r(a+r\cos u)^2 \cos u$$

Since $\sqrt{EG-F^2} = \sqrt{r^2(a+r\cos u)^2} = r(a+r\cos u)$, we then conclude that

$$e = r$$

$$f = 0$$

$$g = (a+r\cos u)\cos u$$

Therefore, $K = \dfrac{eg-f^2}{EG-F^2} = \dfrac{r(a+r\cos u)\cos u}{r^2(a+r\cos u)^2} = \dfrac{\cos u}{r(a+r\cos u)}$

$$H = \frac{1}{2}\frac{eG-2fF+gE}{EG-F^2} = \frac{1}{2}\frac{r(a+r\cos u)^2 - 0 + r^2(a+r\cos u)\cos u}{r^2(a+r\cos u)^2} = \frac{1}{2}\frac{(a+r\cos u)+r\cos u}{r(a+r\cos u)} = \frac{1}{2}\frac{a+2r\cos u}{r(a+r\cos u)}$$

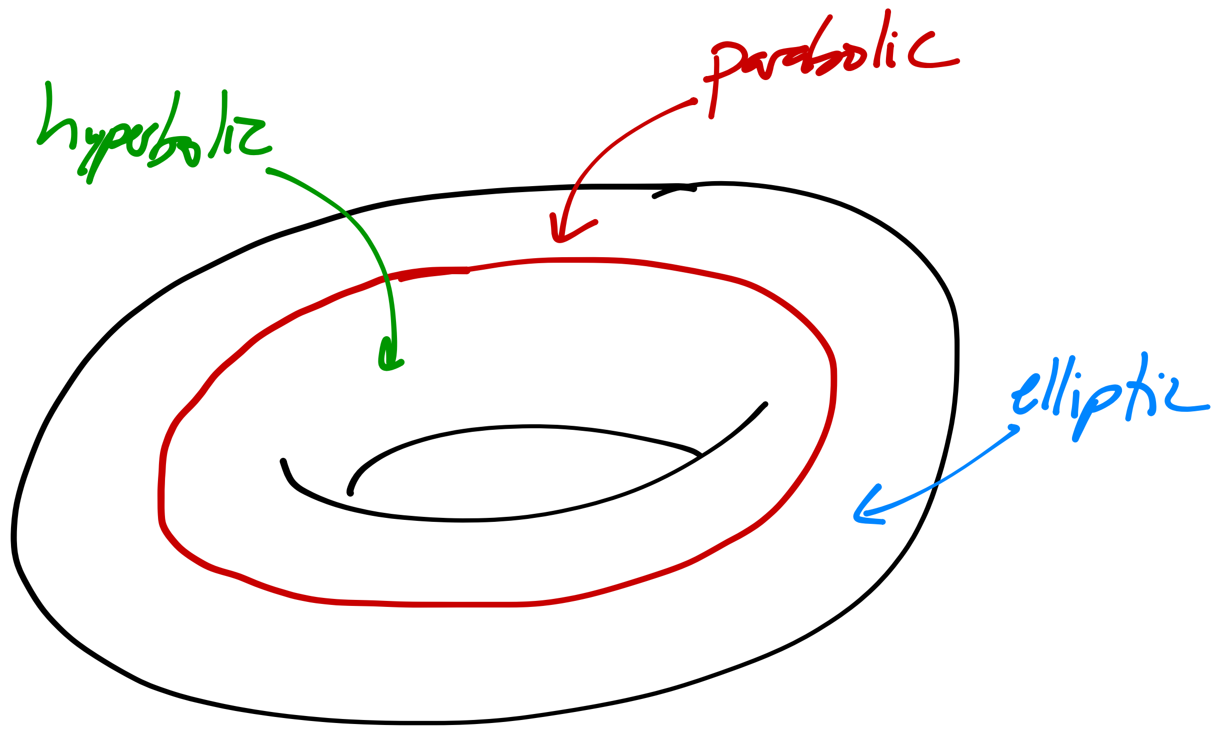
So where are the elliptic, parabolic, & hyperbolic points on the torus?

Well, $a>r>0$, so $(a+r\cos u)>0$ & hence

$$K<0 \iff \cos u<0 \iff u\in(\pi/2, 3\pi/2)$$

$$K=0 \iff \cos u=0 \iff u=\pi/2 \text{ or } 3\pi/2$$

$$K>0 \iff \cos u>0 \iff u\in(0,\pi/2)\cup(3\pi/2, 2\pi)$$



Not really a coincidence that there's a balance b/w hyperbolic & elliptic points.