Math 474: Day 27

Prop: If Σ is connected & umbilic everywhere, then Σ is part of a plane or of a sphere.

Proof: Choose local coords $\vec{x}(u,v)$. If $\vec{w} = a_1\vec{x}_u + b_1\vec{x}_v$, then

$$dN_p(\vec{w}) = \lambda(p)\,\vec{w}. \qquad (*)$$

Claim: $\lambda \equiv \text{const}$.

Pf: Since dN_p is given by $\vec{x}_u \mapsto N_u$ & $\vec{x}_v \mapsto \vec{N}_v$, then we know that $(*)$ becomes

$$a_1\vec{N}_u + a_2\vec{N}_v = \lambda a_1\vec{x}_u + \lambda a_2\vec{x}_v$$

for any a_1, a_2, meaning that $\vec{N}_u = \lambda\vec{x}_u$ & $\vec{N}_v = \lambda\vec{x}_v$, which implies

$$\vec{N}_{uv} = \lambda_v\vec{x}_u + \lambda\vec{x}_{uv}$$
$$\vec{N}_{vu} = \lambda_u\vec{x}_v + \lambda\vec{x}_{vu}$$

But then $\lambda_v\vec{x}_u = \lambda_u\vec{x}_v \Rightarrow \lambda_v\vec{x}_u - \lambda_u\vec{x}_v = 0$. Since $\vec{x}_u$ & $\vec{x}_v$ are lin'ly indep't, this means $\lambda_u = 0$ & $\lambda_v = 0$, so

λ is locally const $\Rightarrow \lambda$ const.

Now, since λ is const., either $\lambda = 0$ or $\lambda \neq 0$. If $\lambda = 0$, then $d\vec{N} \equiv 0$, so $\vec{N}$ is constant. Then for any

$\vec{x}(u,v) \in \Sigma$, we have

$$\frac{\partial}{\partial u}\langle \vec{x}(u,v), \vec{N}\rangle = \langle \vec{x}_u, N\rangle + \langle \vec{x}, N_u\rangle = 0 + 0 = 0$$

& likewise when differentiating w.r.t. v, so $\langle \vec{x}, \vec{N}\rangle \equiv \text{const}$, so $\vec{x}$ is on a plane.

OTOH, if $\lambda \neq 0$, let $\vec{y}(u,v) = \vec{x}(u,v) - \frac{1}{\lambda}\vec{N}(u,v)$ so that

$$\vec{y}_u = \vec{x}_u - \tfrac{1}{\lambda}\vec{N}_u = 0$$
$$\vec{y}_v = \vec{x}_v - \tfrac{1}{\lambda}\vec{N}_v = 0$$

So $\vec{y}$ is constant. Moreover,

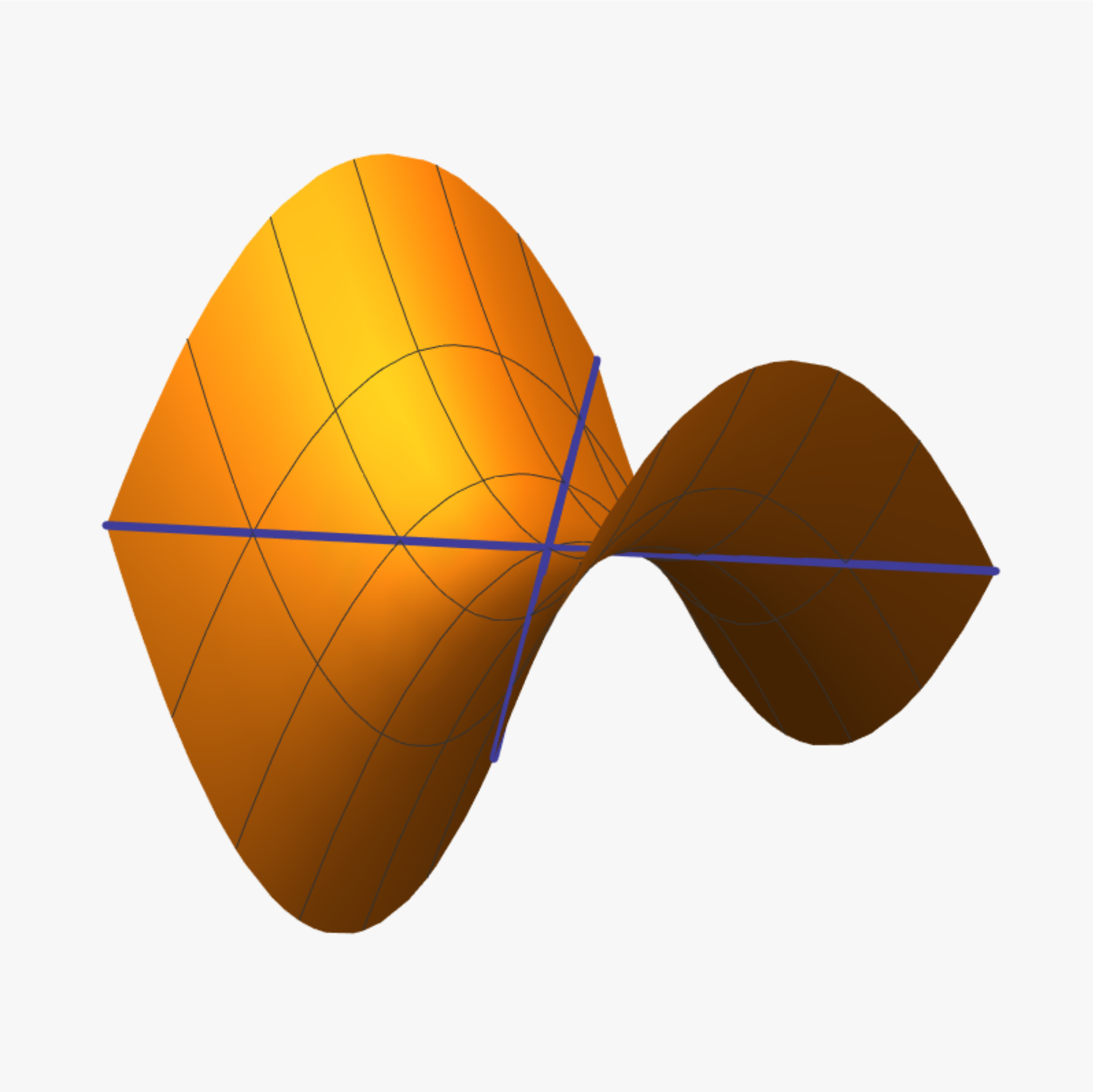
$$|\vec{x}(u,v) - \vec{y}|^2 = \left|\tfrac{1}{\lambda}\vec{N}(u,v)\right|^2 = \frac{1}{\lambda^2}$$

so $\vec{x}(u,v)$ is on the sphere of radius $\frac{1}{\lambda^2}$ centered at $\vec{y}$. ∎

Def: Given $p \in \Sigma$, an *asymptotic direction* of Σ at p is one with normal curvature 0. An *asymptotic curve* has all tangents in asymptotic directions.

Note that elliptic points have no asymptotic directions, but hyperbolic points have two (this is basically the Intermediate Value Theorem on the circle of directions)

Ex:



Computing with II_p

Recall that $d\vec{N}_p : T_p\Sigma \to T_p\Sigma$ maps $\vec{x}_u \mapsto \vec{N}_u$ & $\vec{x}_v \mapsto \vec{N}_v$. It would be nice to write $d\vec{N}_p$ as a matrix in the $\{\vec{x}_u, \vec{x}_v\}$ basis.

It's certainly the case that

$$\vec{N}_u = a_{11}\vec{x}_u + a_{21}\vec{x}_v$$
$$\vec{N}_v = a_{12}\vec{x}_u + a_{22}\vec{x}_v$$

so (by linearity), for $\vec{w} = w_1\vec{x}_u + w_2\vec{x}_v \in T_p\Sigma$,

$$d\vec{N}_p \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

But now let's look at this a different way:

$$\mathrm{II}_p(\vec{w},\vec{w}) = -\langle d\vec{N}_p(\vec{w}), \vec{w} \rangle_{I_p} = -\langle w_1\vec{N}_u + w_2\vec{N}_v,\ w_1\vec{x}_u + w_2\vec{x}_v \rangle_{I_p} = (-\langle \vec{N}_u, \vec{x}_u \rangle) w_1^2 + 2w_1w_2(-\langle \vec{N}_v, \vec{x}_u \rangle = -\langle \vec{N}_u, \vec{x}_v \rangle) + (-\langle \vec{N}_v, \vec{x}_v \rangle) w_2^2$$

(Remember that we showed $\langle \vec{N}_u, \vec{x}_v \rangle = \langle \vec{N}_v, \vec{x}_u \rangle$ before).

These 3 inner products have names:

$$e = -\langle \vec{N}_u, \vec{x}_u \rangle$$
$$f = -\langle \vec{N}_u, \vec{x}_v \rangle = -\langle \vec{N}_v, \vec{x}_u \rangle$$
$$g = -\langle \vec{N}_v, \vec{x}_v \rangle$$

(Shifrin calls these l, m, n for some crazy reason)

In fact, since $\langle \vec{N}, \vec{x}_u \rangle = 0 = \langle \vec{N}, \vec{x}_v \rangle$, we also have

$$e = \langle \vec{N}, \vec{x}_{uu} \rangle_{I_p}$$

$$f = \langle \vec{N}, \vec{x}_{uv} \rangle_{I_p} = \langle \vec{N}, \vec{x}_{vu} \rangle_{I_p}$$

$$g = \langle \vec{N}, \vec{x}_{vv} \rangle_{I_p}$$

Now, remember I_p was given by

$$E = \langle \vec{x}_u, \vec{x}_u \rangle_{I_p}$$

$$F = \langle \vec{x}_u, \vec{x}_v \rangle_{I_p}$$

$$G = \langle \vec{x}_v, \vec{x}_v \rangle_{I_p}$$

So we can compute

$$-f = \langle \vec{N}_u, \vec{x}_v \rangle = \langle a_{11}\vec{x}_u + a_{21}\vec{x}_v, \vec{x}_v \rangle_{I_p} = a_{11} \langle \vec{x}_u, \vec{x}_v \rangle_{I_p} + a_{21} \langle \vec{x}_v, \vec{x}_v \rangle_{I_p} = a_{11}F + a_{21}G$$

$$-f = \langle \vec{N}_v, \vec{x}_u \rangle = a_{12}E + a_{22}F$$

$$-e = \langle \vec{N}_u, \vec{x}_u \rangle = a_{11}E + a_{21}F$$

$$-g = \langle \vec{N}_v, \vec{x}_v \rangle = a_{12}F + a_{22}G$$

or, in matrix language

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

$$= \frac{1}{EG-F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

& hence

$$\underbrace{a_{11} = \frac{fF-eG}{EG-F^2} \qquad a_{12} = \frac{gF-fG}{EG-F^2} \qquad a_{21} = \frac{eF-fE}{EG-F^2} \qquad a_{22} = \frac{fF-gE}{EG-F^2}}_{\text{Weingarten Equations}}$$