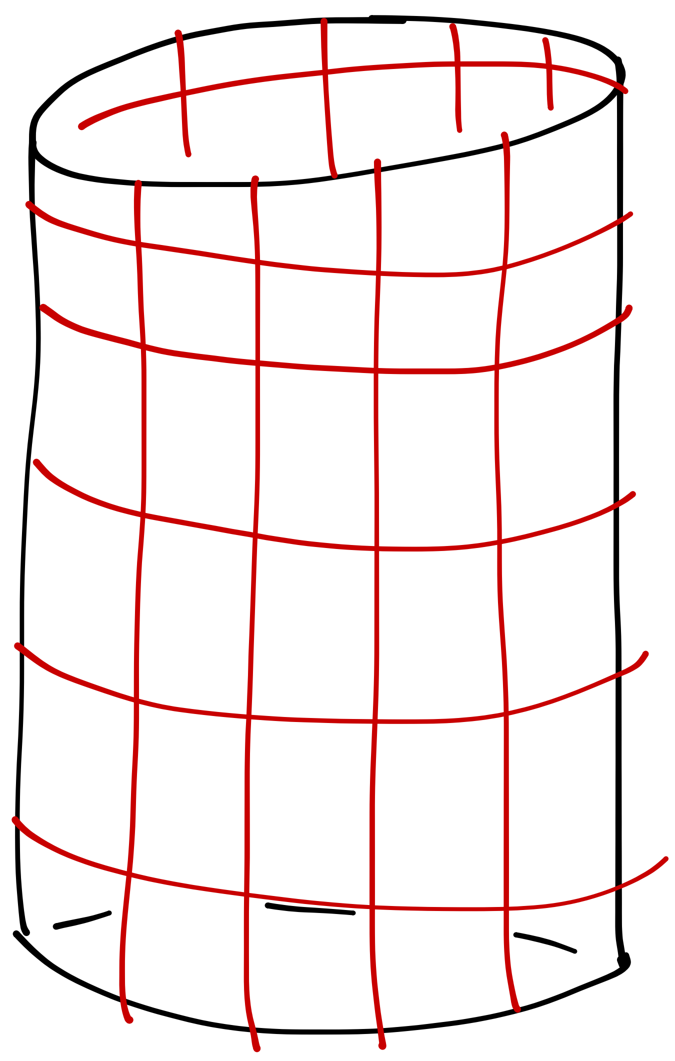
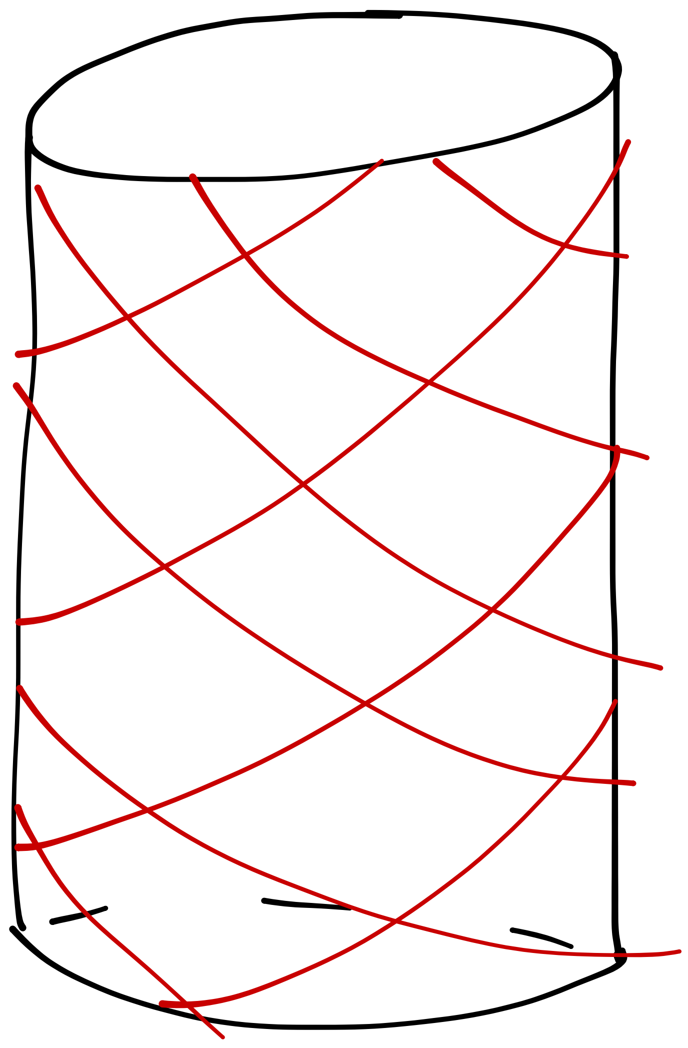
# Math 474: Day 26

**Def:** If $\alpha(s) \subseteq \Sigma$ is a curve on $\Sigma$ so that $\alpha'(s)$ is in a principal direction for $\Sigma$ at $\alpha(s)$, then $\alpha$ is a line of curvature.

**Ex:**



Lines of curvature



Not lines of curvature

**Prop:** $\alpha$ is a line of curvature $\iff$ $N'(t) = \lambda(t)\alpha'(t)$ for some differentiable function $\lambda(t)$. In fact, $-\lambda(t)$ turns out to be the curvature along $\alpha$.

**Proof:** If $\alpha(t)$ is a line of curvature, then

$$N'(t) = dN(\alpha'(t)) = \lambda(t)\alpha'(t)$$

since $\alpha'(t)$ must be an eigenvector of $dN$. Moreover, the eigenvalue $\lambda(t)$ is minus the principal curvature in that direction. Conversely, since the eigenvectors of $dN$ are the principal directions, if $N'(t) = \lambda(t)\alpha'(t)$, then $\alpha$ is a line of curvature. ∎

Now, since the eigenvalues of $dN$ are the normal curvatures, we know that any elementary symmetric polynomials of the eigenvalues will be invariant under change of basis (which is to say, under different choices of local coords.). So the geometric invariants are the determinant & the trace:

**Def:** The Gaussian curvature of $\Sigma$ is given by $K = \kappa_1\kappa_2$ (product of principal curvatures)

and the mean curvature is $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ (average of principal curvatures)

These are the determinant & half the trace of $dN_p$ as a map $T_p\Sigma \to T_p\Sigma$.

A point $p \in \Sigma$ is called:

- elliptic if $\det dN_p = K > 0$
- hyperbolic if $\det dN_p = K < 0$
- parabolic if $\det dN_p = K = 0$ but $dN_p \neq 0$
- planar if $dN_p = 0$.

Ex:

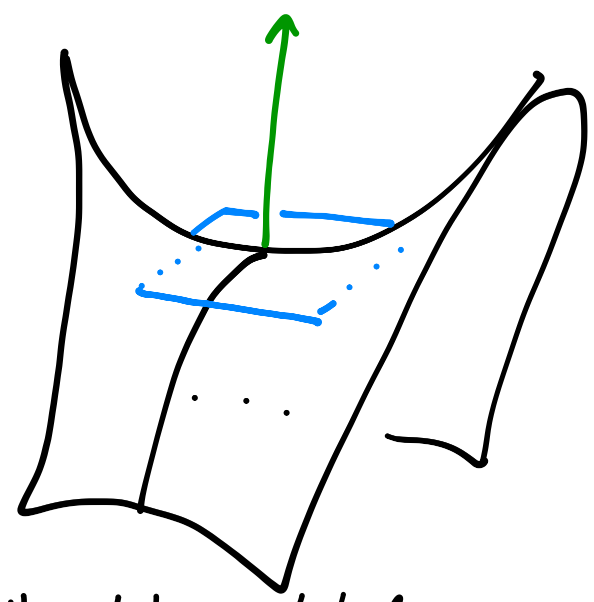
round sphere

$dN_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\kappa_1 = \kappa_2 = -1$,

$K = \kappa_1 \cdot \kappa_2 = 1$, $H = -1$

elliptic

Ex:



Hyperbolic paraboloid

$\vec{x}(u,v) = (u, v, v^2 - u^2)$

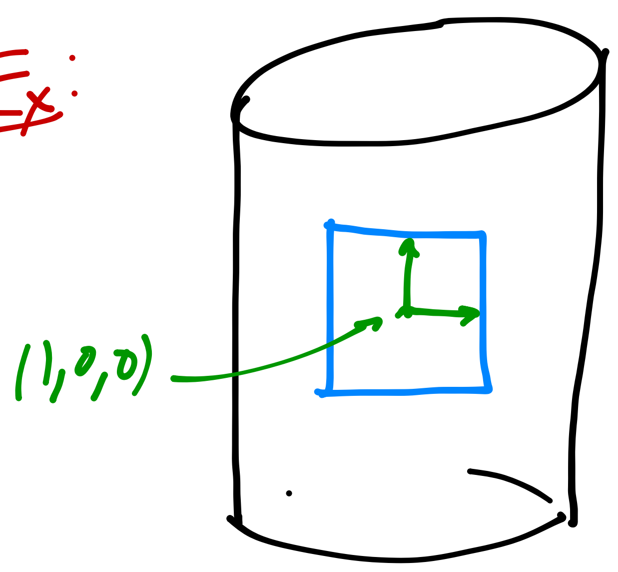
$dN_p(a,b,0) = (2a, -2b, 0)$

$\kappa_1 = -2$, $\kappa_2 = 2$

$K = -4$, $H = 0$

hyperbolic

Ex:




cylinder $\vec{x}(z,\theta) = (\cos\theta, \sin\theta, z)$

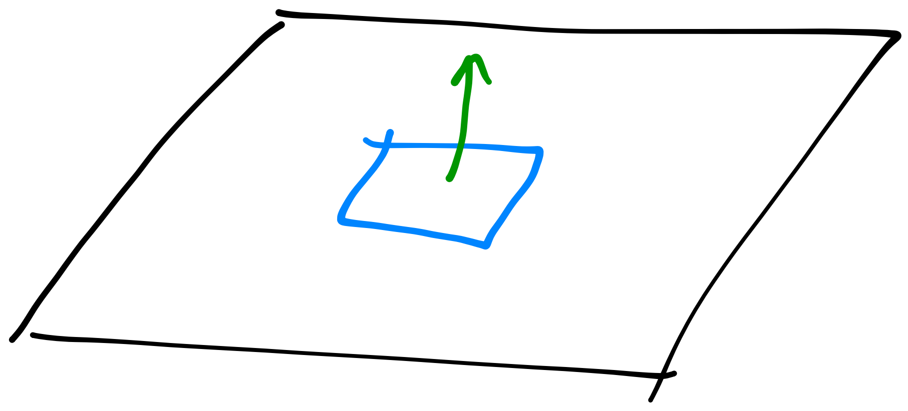
$dN_p(0,a,b) = (0,a,0)$

$\kappa_1 = -1$, $\kappa_2 = 0$

$K = 0$, $H = -\frac{1}{2}$

parabolic

Ex:



plane

$dN_p = 0$

$K = 0$, $H = 0$

planar

Note for later: Surfaces w/ $H = 0$ are called minimal surfaces. They are critical points of area.

Def: If $\kappa_1 = \kappa_2$ at a point, the point is called an umbilic point.

(Maybe) surprising fact:

Prop: If $\Sigma$ is connected & umbilic everywhere, then $\Sigma$ is part of a plane or of a sphere.