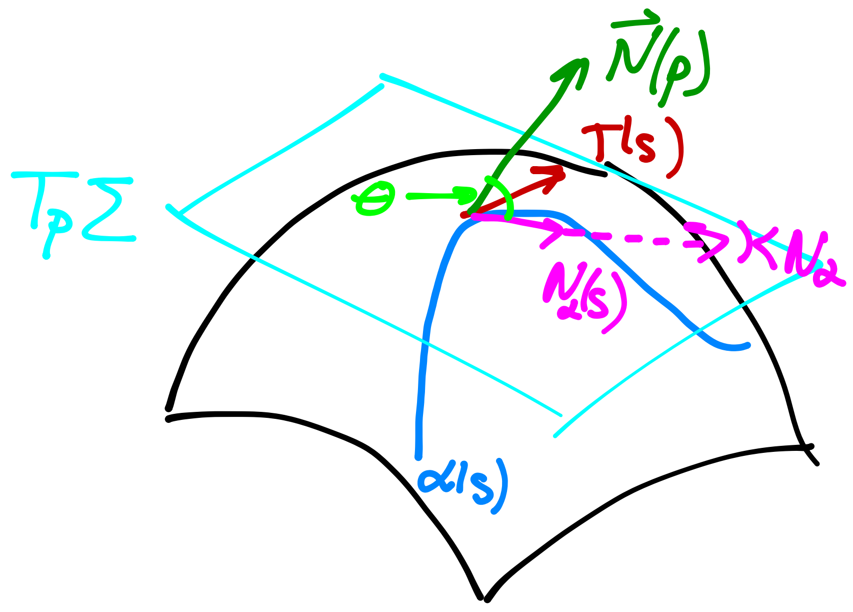
Math 474: Day 25

Recall the definition of the second fundamental form

$$\mathrm{II}_p(\vec{w}_1,\vec{w}_2) = \langle -d\vec{N}_p(\vec{w}_1), \vec{w}_2 \rangle_{\mathrm{I}_p}$$

What does it mean? Well, suppose we have a curve $\alpha(s)$ on the surface Σ.



Then $T(s) = \alpha'(s) \in T_p\Sigma$, but of course the normal vector $N_\alpha(s)$ doesn't have to lie in the tangent plane, & usually doesn't.

Def: The **normal curvature** of $\alpha(s)$ at s_0 is given by

$$\kappa_n(s_0) = \langle \underbrace{\kappa(s_0) N_\alpha(s_0)}_{\text{curve normal}}, \underbrace{\vec{N}(\alpha(s_0))}_{\text{surface normal}} \rangle$$

Prop: $\kappa_n(s_0) = \mathrm{II}_{\alpha(s_0)}(\alpha'(s_0), \alpha'(s_0))$

Proof: By definition, $\kappa_n(s_0) = \langle \alpha''(s_0), \vec{N}(\alpha(s_0)) \rangle$. Since $\alpha'(s_0) \in T_{\alpha(s_0)}\Sigma \perp \vec{N}(\alpha(s_0))$, we know $\langle \alpha'(s_0), \vec{N}(\alpha(s_0)) \rangle = 0$, so

$$0 = \frac{d}{ds}\langle \alpha'(s), \vec{N}(\alpha(s)) \rangle = \langle \alpha''(s), N(\alpha(s)) \rangle + \langle \alpha'(s), \frac{d}{ds}(N(\alpha(s))) \rangle$$

and hence

$$\kappa_n(s_0) = -\langle \alpha'(s_0), \frac{d}{ds}\Big|_{s=s_0}(N(\alpha(s))) \rangle$$

Now, $\frac{d}{ds}\Big|_{s=s_0}(N(\alpha(s)))$ is the change in $\vec{N}$ as the point $p = \alpha(s_0)$ moves in the direction of $\alpha'(s_0)$. But that's exactly the definition of $d\vec{N}_{\alpha(s_0)}(\alpha'(s_0))$, so

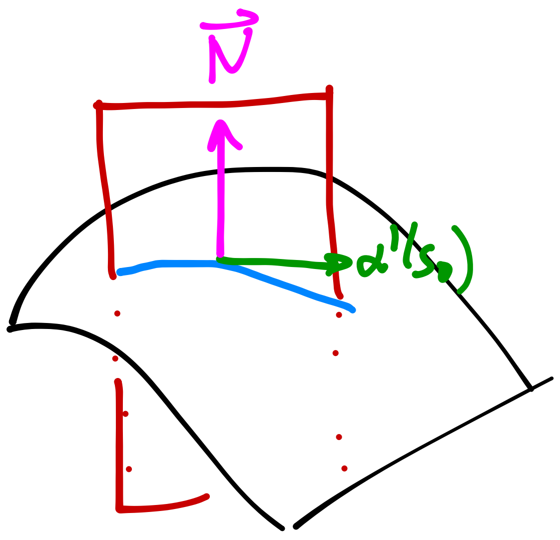
$$\kappa_n(s_0) = -\langle \alpha'(s_0), d\vec{N}_{\alpha(s_0)}(\alpha'(s_0)) \rangle = \mathrm{II}_p(\alpha'(s_0), \alpha'(s_0)). \qquad \blacksquare$$

Note that, using an algebraic trick (which is a sort of "differentiation by parts") we shifted a derivative from α'' to N, which meant we computed a second derivative (κ_n) in terms of first derivatives (α', $d\vec{N}$).

But this means:

Meusnier's Thm: All curves in Σ thru a point p with the same tangent vector at p have the same normal curvature at p.

Of course, one such curve will have the least possible curvature κ:

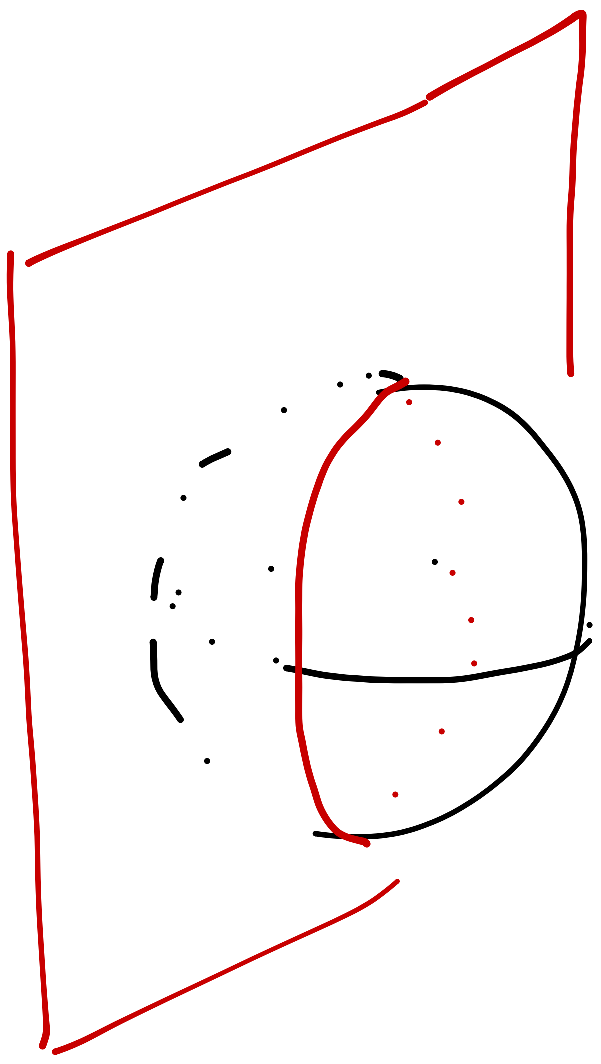


The intersection of the plane containing $\vec{N}(\alpha(s_0))$ & $\alpha'(s_0)$ with Σ is called the normal section of Σ in direction $\alpha'(s_0)$.

Now, the curvature of the normal section = the normal curvature of any curve in Σ w/ the same tangent at $p = \alpha'(s_0)$

$$= \mathrm{II}_p(\alpha'(s_0), \alpha'(s_0))$$ (so long as $\alpha'(s_0)$ is a unit vector)

Ex: We can compute II_p easily using normal sections. On S^2, the normal sections are unit circles, so



$$\mathrm{II}_p(\vec{v}, \vec{v}) = -1 = -\mathrm{I}_p(\vec{v}, \vec{v})$$

for all unit tangent vectors $\vec{v}$.

Q: Does this determine II_p?

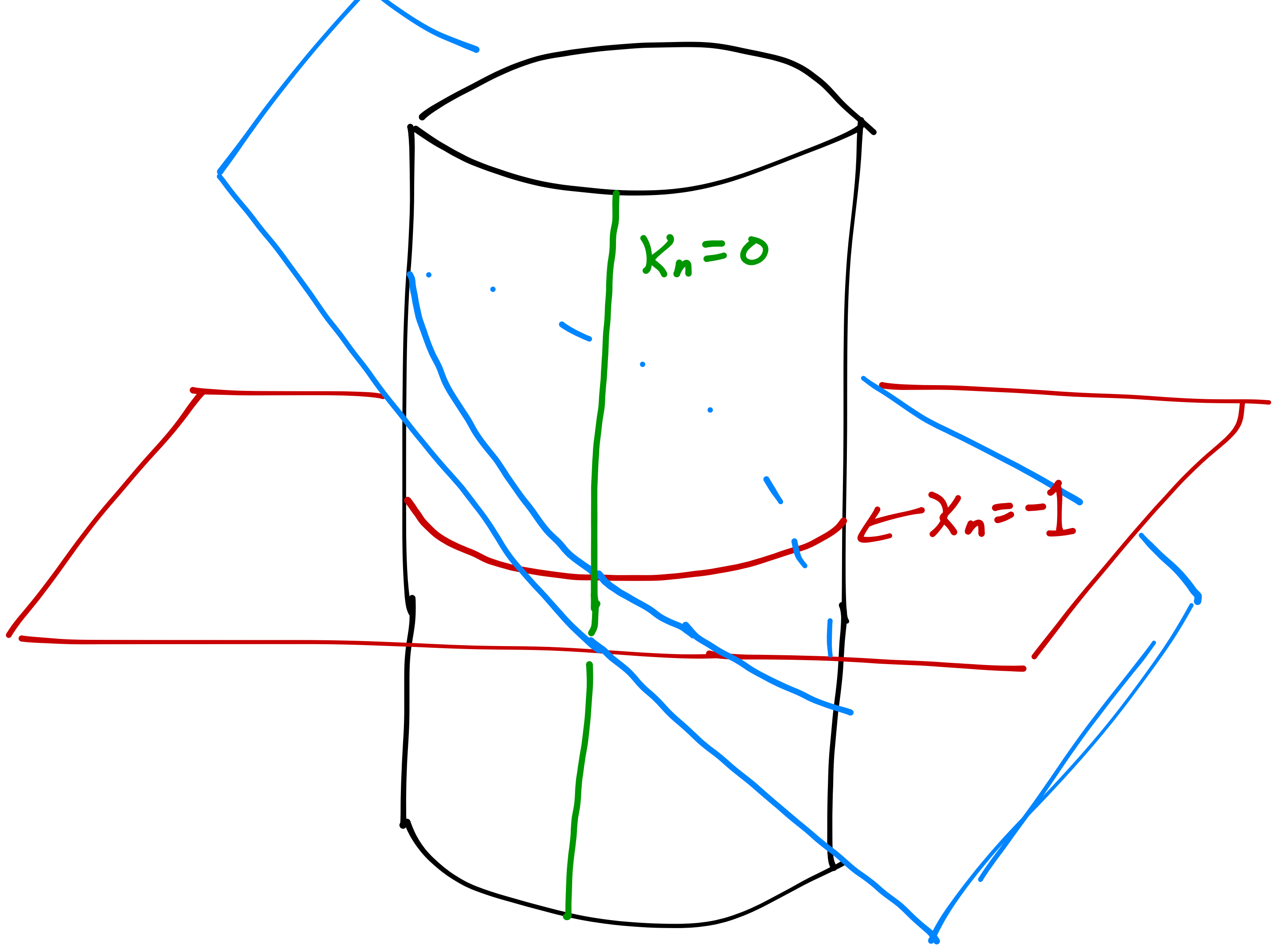
Polarization Trick: For a quadratic form Q,

$$Q(\vec{v}, \vec{w}) = \frac{1}{2}\left\{ Q(\vec{v}+\vec{w}, \vec{v}+\vec{w}) - Q(\vec{v}, \vec{v}) - Q(\vec{w}, \vec{w}) \right),$$

which follows from bilinearity in expanding $Q(\vec{v}+\vec{w}, \vec{v}+\vec{w})$.

Therefore, on the sphere we have $\mathrm{II}_p(\vec{v}, \vec{w}) = -\mathrm{I}_p(\vec{v}, \vec{w})$ for all $\vec{v}, \vec{w}$ & hence the matrix associated to $-\mathrm{II}_p$ is the identity matrix.

(which we also could have seen since $d\vec{N}_p$ is the identity).



On the cylinder, κ_n varies from -1 to 0, depending on the choice of direction in $T_p\Sigma$

But notice that the direction of maximum (0) & minimum (-1) normal curvatures are perpendicular

This is not an accident...

Prop: Given 2 quadratic forms Q & P on $\mathbb{R}^2$, there is a basis $\vec{v}_1, \vec{v}_2$ of $\mathbb{R}^2$ so that

$$\langle \vec{v}_i, \vec{v}_j \rangle_P = \delta_{ij} \quad \text{or} \quad P(v_i, v_j) = \delta_{ij}$$

(the basis is ON w.r.t. P) and

$$Q(\vec{v}_1, \vec{v}_1) = \lambda_1 \quad \& \quad Q(\vec{v}_2, \vec{v}_2) = \lambda_2$$

are the max & min values of Q on the "unit circle" $\{\cos\theta \vec{v}_1 + \sin\theta \vec{v}_2\}$.

Though we won't prove it, it also turns out to be the case that $\vec{v}_1$ & $\vec{v}_2$ are eigenvectors for dN_p. So, in this basis, if

$$d\vec{N}_p(\vec{v}_1) = -\kappa_1 \vec{v}_1 \qquad \mathrm{II}_p(\vec{v}_1, \vec{v}_1) = \kappa_1$$
then
$$d\vec{N}_p(\vec{v}_2) = -\kappa_2 \vec{v}_2 \qquad \mathrm{II}_p(\vec{v}_2, \vec{v}_2) = \kappa_2$$

are the maximal & minimal normal curvatures at p.

Def: The maximum κ_1 & minimum κ_2 normal curvatures are called the *principal curvatures* of Σ at p. The directions $\vec{v}_1$ & $\vec{v}_2$ are called *principal directions*

We can define some special coordinate lines on our surface by:

Def: If $\alpha(s) \subseteq \Sigma$ is a curve on Σ so that $\alpha'(s)$ is in a principal direction for Σ at $\alpha(s)$, then α is a *line of curvature*.