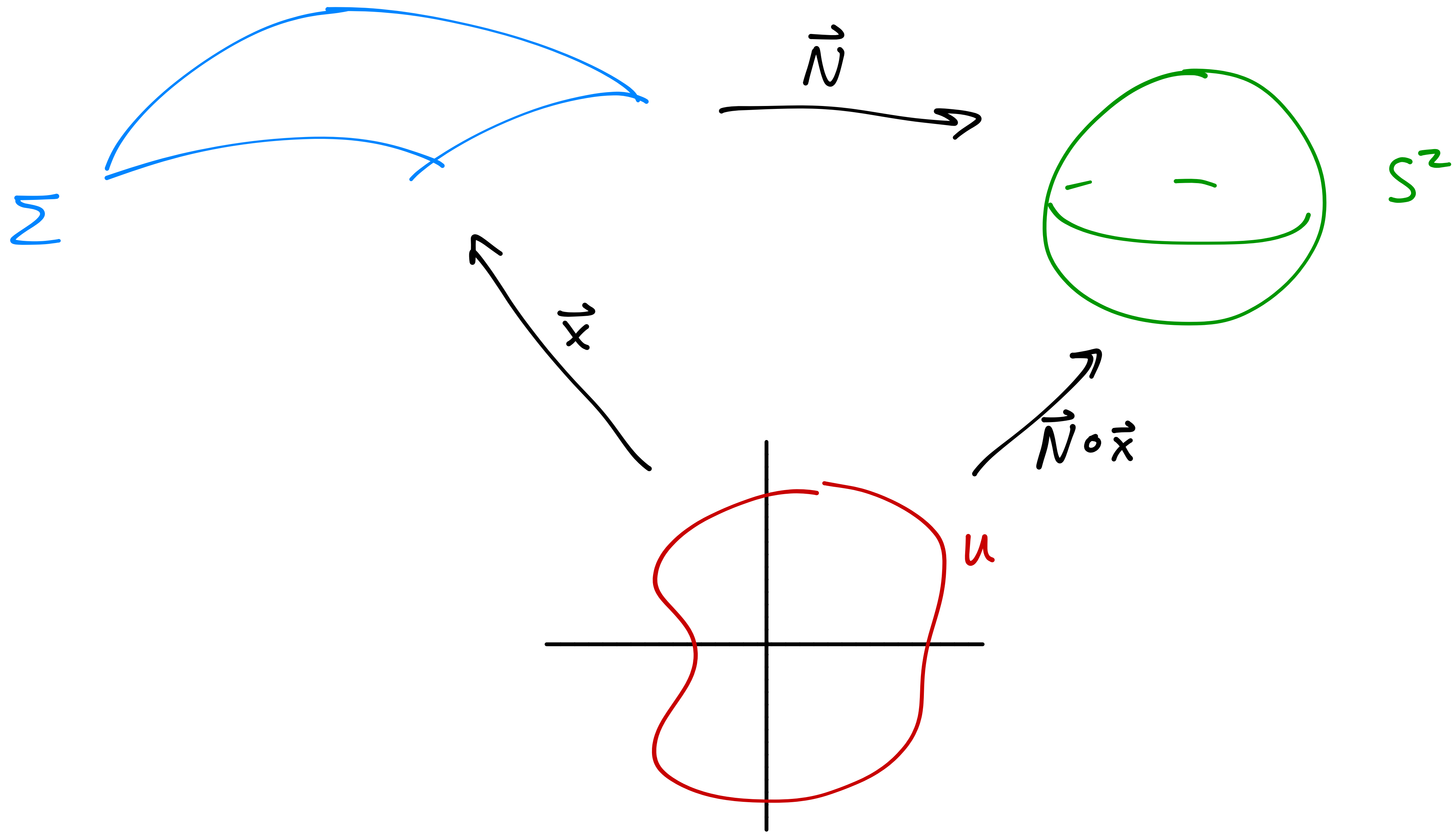
Math 474: Day 24

Of course, your first question should be: why the hell would we do such a confusing thing by thinking of $d\vec{N}_p$ as a map from $T_p\Sigma$ to itself.

Answer: Eigenvalues! (More to come...)

As for computing, the idea is to use local coords. on Σ as much as possible.



Then $\vec{N}\circ\vec{x}$ defines local coords. on S^2 &, w.r.t. these coords., $N(u,v)=(u,v)$ & so $d\vec{N}_p = Id$. But of course you have to be careful: on Σ we're using the basis $\{\vec{x}_u, \vec{x}_v\}$ for $T_p\Sigma$, whereas on S^2 we're in the basis $\vec{N}_u, \vec{N}_v$.

So as a map $T_p\Sigma \to T_p\Sigma$, $d\vec{N}_p$ is given by:
$$\vec{x}_u \mapsto \vec{N}_u$$
$$\vec{x}_v \mapsto \vec{N}_v$$

Ex: $\mathbb{R}^2$: $\vec{x}(u,v)=(u,v,0)$, $N(u,v)=(0,0,1)$, so

$\vec{x}_u=(1,0,0)$ $\quad$ $\vec{N}_u=(0,0,0)$

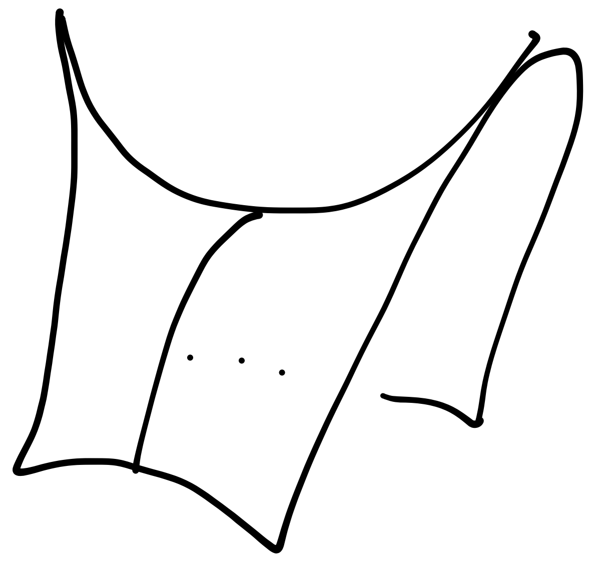
$\vec{x}_v=(0,1,0)$ $\quad$ $\vec{N}_v=(0,0,0)$, $\quad$ & so $d\vec{N}_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $p\in\mathbb{R}^2$.

Ex: The unit sphere. $\vec{x}(u,v)=(u,v,\sqrt{1-u^2-v^2}) \Rightarrow \vec{x}_u=\left(1,0,\frac{-u}{\sqrt{1-u^2-v^2}}\right)$ & $\vec{x}_v=\left(0,1,\frac{-v}{\sqrt{1-u^2-v^2}}\right)$

Hence, $\vec{x}_u\times\vec{x}_v = \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right) = \frac{1}{\sqrt{1-u^2-v^2}}(u,v,\sqrt{1-u^2-v^2})$, so $\vec{N}=\frac{\vec{x}_u\times\vec{x}_v}{|\vec{x}_u\times\vec{x}_v|} = (u,v,\sqrt{1-u^2-v^2}) = \vec{x}(u,v)$.

Therefore $\vec{N}_u=\vec{x}_u$ & $\vec{N}_v=\vec{x}_v$, so $d\vec{N}_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Ex: The hyperbolic Paraboloid



$$\vec{X}(u,v) = (u, v, v^2 - u^2)$$

Then $\vec{X}_u = (1, 0, -2u)$, $\vec{X}_v = (0, 1, 2v)$

$$\vec{X}_u \times \vec{X}_v = (2u, -2v, 1) \Rightarrow |\vec{X}_u \times \vec{X}_v| = \sqrt{4u^2+4v^2+1} = 2\sqrt{u^2+v^2+1/4}, \text{ so}$$

$$\vec{N} = \left(\frac{u}{\sqrt{u^2+v^2+1/4}}, \frac{-v}{\sqrt{u^2+v^2+1/4}}, \frac{1}{2\sqrt{u^2+v^2+1/4}}\right) = \frac{1}{\sqrt{u^2+v^2+1/4}}(u, -v, 1/2) \quad \& \text{ hence}$$

$$\vec{N}_u = \frac{-u}{(u^2+v^2+1/4)^{3/2}}(u, -v, 1/2) + \frac{1}{\sqrt{u^2+v^2+1/4}}(1,0,0) = \frac{1}{\sqrt{u^2+v^2+1/4}}\left(1 - \frac{u^2}{u^2+v^2+1/4}, \frac{uv}{u^2+v^2+1/4}, \frac{-u}{2(u^2+v^2+1/4)}\right)$$

$$\& \quad \vec{N}_v = \frac{1}{\sqrt{u^2+v^2+1/4}}\left(\frac{-uv}{u^2+v^2+1/4}, -1 - \frac{v^2}{u^2+v^2+1/4}, \frac{-v}{2(u^2+v^2+1/4)}\right)$$

In particular, at the point $(u,v) = (0,0)$, we get

$$\vec{X}_u = (1,0,0) \qquad \vec{N}_u = (2,0,0) = 2\vec{X}_u$$

$$\vec{X}_v = (0,1,0) \qquad \vec{N}_v = (0,-2,0) = -2\vec{X}_v, \text{ so, at this point}$$

$$d\vec{N}_p = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Notice that in all 3 examples, $d\vec{N}_p$ is a symmetric matrix. In fact:

Prop: For any vectors $\vec{w}_1, \vec{w}_2 \in T_p\Sigma$, we have

$$\langle \vec{w}_1, d\vec{N}_p \vec{w}_2 \rangle_p = \langle d\vec{N}_p \vec{w}_1, \vec{w}_2 \rangle_p$$

(where $\langle\, , \rangle_p$ is the inner product given by I_p).

Proof: Given basis $\vec{X}_u, \vec{X}_v$ for $T_p\Sigma$, it suffices to show that $\langle \vec{X}_u, d\vec{N}_p \vec{X}_v \rangle = \langle d\vec{N}_p \vec{X}_u, \vec{X}_v \rangle$ as vectors in $\mathbb{R}^3$.

But $d\vec{N}_p \vec{X}_u = \vec{N}_u$ & $d\vec{N}_p \vec{X}_v = \vec{N}_v$, so it suffices to show $\langle \vec{X}_u, \vec{N}_v \rangle = \langle \vec{N}_u, \vec{X}_v \rangle$.

But differentiating $\langle \vec{N}, \vec{X}_u \rangle = 0$ & $\langle \vec{N}, \vec{X}_v \rangle = 0$ gives

$$\frac{\partial}{\partial v}\langle \vec{N}, \vec{X}_u \rangle = \langle \vec{N}_v, \vec{X}_u \rangle + \langle \vec{N}, \vec{X}_{uv} \rangle = 0 \quad \& \quad \frac{\partial}{\partial u}\langle \vec{N}, \vec{X}_v \rangle = \langle \vec{N}_u, \vec{X}_v \rangle + \langle \vec{N}, \vec{X}_{vu} \rangle = 0.$$

Since $\vec{X}_{uv} = \vec{X}_{vu}$, this shows that $\langle \vec{N}_v, \vec{X}_u \rangle = \langle \vec{N}_u, \vec{X}_v \rangle$, as desired. ∎

Now, we have to be a bit careful. It's not really true that $d\vec{N}_p$ is a symmetric matrix... rather, it shows that $I_p d\vec{N}_p$ is symmetric

The symmetry from the proposition naturally leads us to define the associated quadratic form:

Def: The quadratic form

$$\mathrm{II}_p(\vec{w}_1, \vec{w}_2) := -\langle d\vec{N}_p \vec{w}_1, \vec{w}_2 \rangle_p$$

is called the **second fundamental form** of the surface Σ at $p \in \Sigma$.