Math 474: Day 23

Def: If $R \subseteq U$ is a bounded region in the parameter plane of a regular surface given by $\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, then

$$\mathrm{Area}(\vec{x}(R)) := \mathrm{Area}(R) = \iint_R |\vec{x}_u \times \vec{x}_v| \, du\, dv$$

Now, we use the very handy identity

$$|\vec{x}_u \times \vec{x}_v|^2 + \langle \vec{x}_u, \vec{x}_v \rangle_p^2 = \|\vec{x}_u\|_p^2 \, \|\vec{x}_v\|^2$$

to write

$$\mathrm{Area}(R) = \iint_R \sqrt{EG - F^2} \, du\, dv$$

where $\sqrt{EG-F^2}$ gets called the element of area. Notice that this quantity is the square root of the determinant of the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ which maps $T_{p_0}\mathbb{R}^2 \to T_{\vec{p}}\Sigma$.

Ex: Using spherical coordinates $\vec{x}(\varphi, \theta) = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$ as before, the area of the sphere should be the area of the region $R = \{(\varphi, \theta) : 0 < \varphi < 2\pi, \ 0 < \theta < \pi\}$ in this sense.

We already computed $E = \sin^2\theta$, $F = 0$, $G = 1$, so

$$\sqrt{EG - F^2} = \sqrt{\sin^2\theta}, \quad \text{so}$$

$$\mathrm{Area}(S^2) = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \sqrt{EG-F^2}\, d\theta\, d\varphi = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \sqrt{\sin^2\theta}\, d\theta\, d\varphi = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} |\sin\theta|\, d\theta\, d\varphi$$

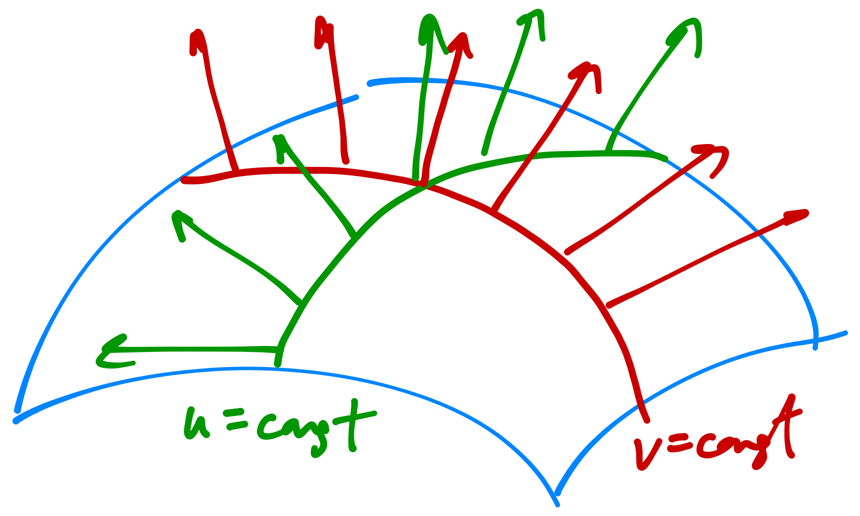
$$= 2\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \sin\theta\, d\theta\, d\varphi = 2\int_{\varphi=0}^{2\pi} 1\, d\varphi = 4\pi,$$ which is indeed the area of the sphere

Notice: This is the same $\sin\theta$ that appears in the integration factor $r^2 \sin\theta\, dr\, d\theta\, d\varphi$ when integrating w.r.t. spherical coords.

The Gauss Map & the Second Fundamental Form

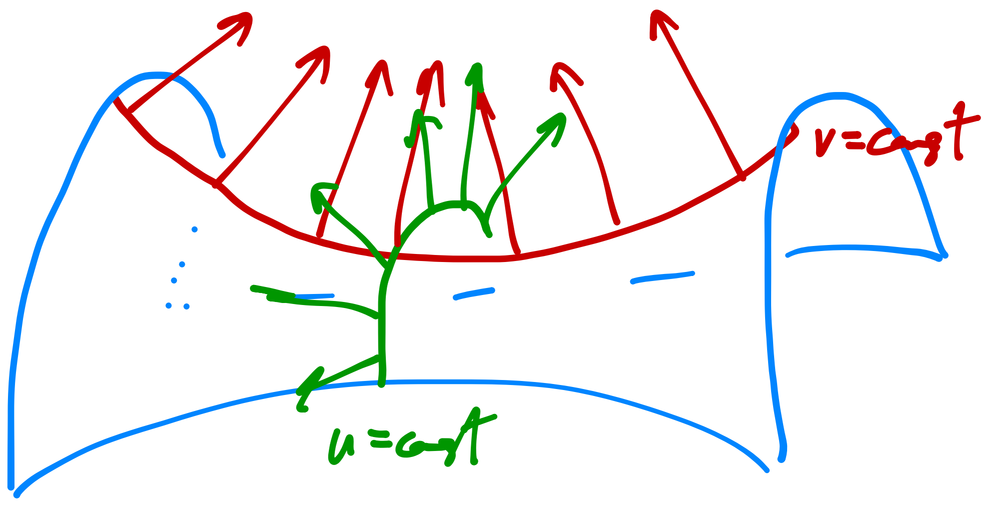
Remember that we defined the curvature of an arclength parametrized curve to be $|\alpha''(s)|$, which measures how fast the normal vector is changing direction.

The idea for surfaces is more or less the same. Again, we want to look at how much the unit normal vector is changing.



In this surface, all the coordinate lines bend the same way, which we can see by looking at the normal vectors.

The goal is to distinguish this scenario from one where the surface bends in opposite directions on different coord. lines



Def: A surface Σ is orientable if there is a differentiable map $\vec{N}: \Sigma \to S^2$ so that $\vec{N}(p) \perp T_p\Sigma$ at every $p \in \Sigma$.

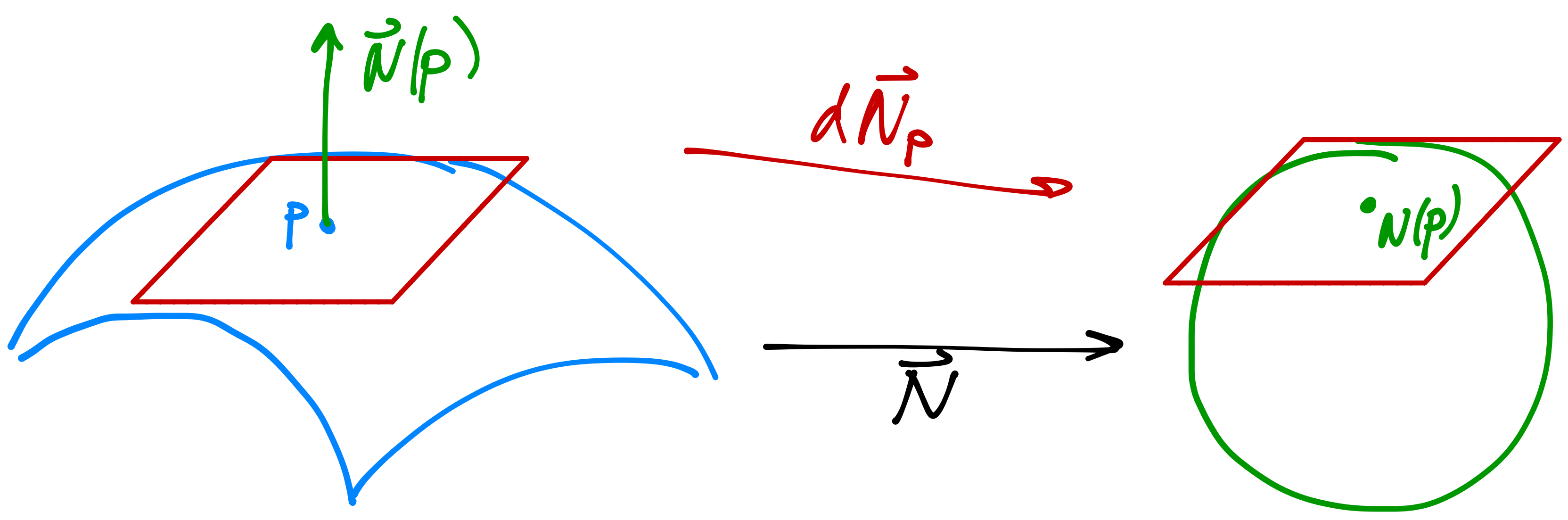
The map $\vec{N}$ is often called the Gauss map.

Given a parametrization $\vec{X}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, we can write $\vec{N}$ explicitly since

$$\vec{N}(p) = \pm \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}$$

DANGER: Not all regular surfaces are orientable. For example, the Möbius strip (see hands-on demo).

Now, here's where things start to get confusing:



Since $\vec{N}: \Sigma \to S^2$, for each $p \in \Sigma$ we have

$$d\vec{N}_p : T_p\Sigma \to T_{\vec{N}(p)}S^2.$$

But $T_{\vec{N}(p)}S^2 \perp \vec{N}(p) \perp T_p\Sigma$, so there is a natural identification $T_{\vec{N}(p)}S^2 \xrightarrow{Id} T_p\Sigma$ and we can think of $d\vec{N}_p$ as a map $d\vec{N}_p : T_p\Sigma \to T_p\Sigma$.

Of course, your first question should be: why the hell would we do such a confusing thing?