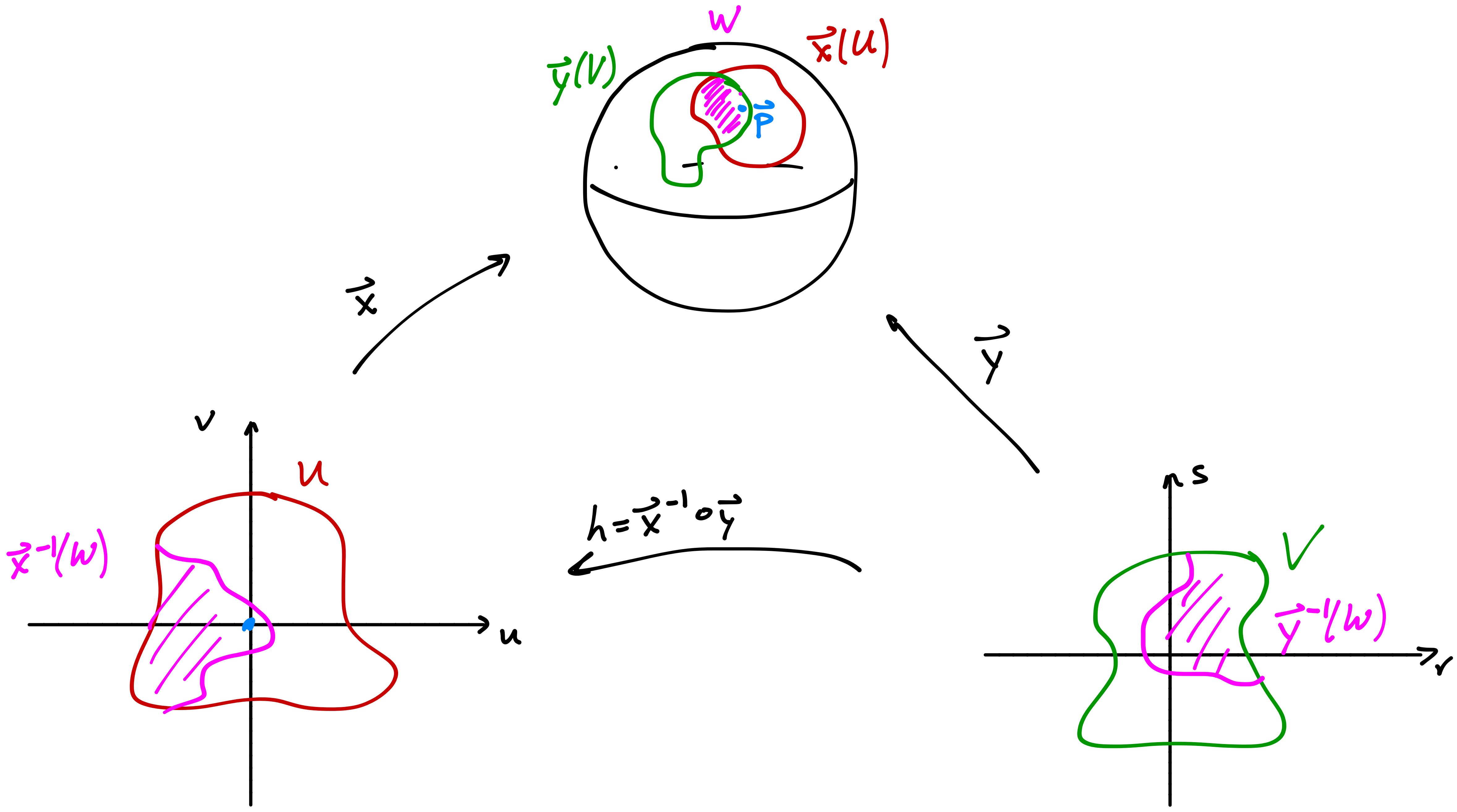
Math 474: Day 19

Prop (Change of Parameters): Let $\vec{p} \in \Sigma$ be a point on a regular surface & let $\vec{x}: U \to \Sigma$ & $\vec{y}: V \to \Sigma$ be two parametrizations w/

$$\vec{p} \in \vec{x}(U) \cap \vec{y}(V) = W.$$

Then the change-of-coordinates map $h = \vec{x}^{-1} \circ \vec{y}: \vec{y}^{-1}(W) \to \vec{x}^{-1}(W)$ is a diffeomorphism, meaning it is differentiable & has differentiable inverse.



So this proposition tells us that any coords. (u,v) on Σ can be written as differentiable functions $u(r,s)$ & $v(r,s)$ of any other local coords. (r,s) on an overlapping chart.

Then if $F(u,v)$ is differentiable w.r.t. u & v, it is also differentiable w.r.t. r & s by the Chain Rule:

$$\frac{\partial F}{\partial r} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial r} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial r}.$$

Ex: Let $\Sigma = S^2$ & suppose F is given in the local coords (y,z) (i.e., $\vec{x}(y,z) = (\sqrt{1-y^2-z^2}, y, z)$). If $F(y,z) = z$, then F is certainly differentiable, since $\frac{\partial F}{\partial y} = 0$ & $\frac{\partial F}{\partial z} = 1$.

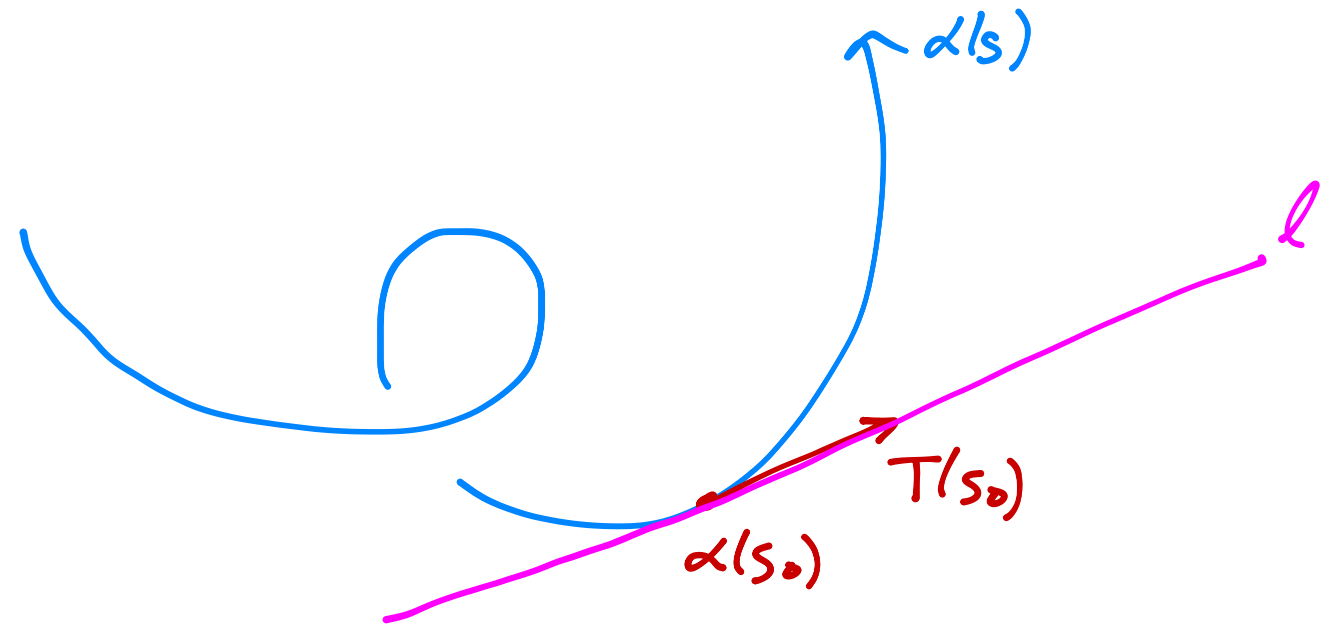
Now, how about in (x,y) coords? Well, $x = \sqrt{1-y^2-z^2}$ & $y = y$, so $\frac{\partial F}{\partial y} = 0$ & $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$

$$= 0 + 1 \cdot \frac{-z}{\sqrt{1-y^2-z^2}} = \frac{-z}{\sqrt{1-y^2-z^2}}$$

The Tangent Plane

Now, we want to consider the tangent plane to a surface parametrized by $\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. The idea is to generalize the notion of a tangent line to a curve $\alpha(s)$.

In general, the line through a point $\vec{p}$ in the direction of a vector $\vec{v}$ is given by $\ell(t) = \vec{p} + t\vec{v}$.



So the tangent line to $\alpha(s)$ at s_0 is

$$\ell(t) = \alpha(s_0) + t\,T(s_0) = \alpha(s_0) + t\,\alpha'(s_0)$$

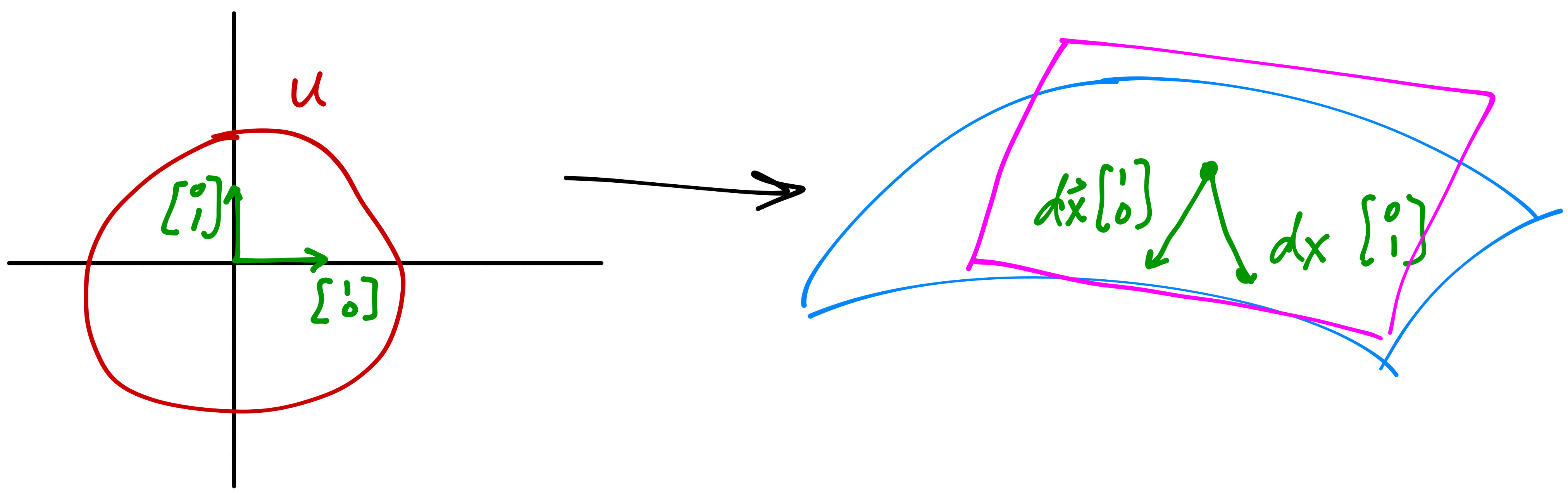
Now, it's slightly better to think of $\alpha'(s_0)$ as the differential of $\alpha : (a,b) \to \mathbb{R}^3$:

$$d\alpha_{s_0} : \mathbb{R} \to \mathbb{R}^3 \quad \text{where} \quad d\alpha_{s_0} = \begin{bmatrix} \frac{\partial x}{\partial s}\big|_{s_0} \\ \frac{\partial y}{\partial s}\big|_{s_0} \\ \frac{\partial z}{\partial s}\big|_{s_0} \end{bmatrix}$$

So then the tangent line ℓ is just the image of $d\alpha_{s_0}$, shifted parallel to itself until it passes through $\alpha(s_0)$:

$$\ell = \alpha(s_0) + \text{Image}(d\alpha_{s_0})$$

This is the idea that works for surfaces:



The tangent plane to Σ at the point $\vec{x}(u_0,v_0)$ is the span of

$$\vec{x}_u\big|_{(u_0,v_0)} = \begin{bmatrix} \frac{\partial x}{\partial u}\big|_{(u_0,v_0)} \\ \frac{\partial y}{\partial u}\big|_{(u_0,v_0)} \\ \frac{\partial z}{\partial u}\big|_{(u_0,v_0)} \end{bmatrix} \quad \& \quad \vec{x}_v\big|_{(u_0,v_0)} = \begin{bmatrix} \frac{\partial x}{\partial v}\big|_{(u_0,v_0)} \\ \frac{\partial y}{\partial v}\big|_{(u_0,v_0)} \\ \frac{\partial z}{\partial v}\big|_{(u_0,v_0)} \end{bmatrix}$$

translated by $\vec{x}(u_0,v_0)$.

Another way to think of it is: this plane is normal to the vector $\vec{n} = \vec{x}_u \times \vec{x}_v$. Since the equation of a plane normal to a vector $\vec{w}$ through a point $\vec{p}$ is given by the equation $\vec{w}\cdot(x,y,z) = \vec{w}\cdot\vec{p}$, we get that the plane is given by the eqn

$$\vec{n}\cdot(x,y,z) = \vec{n}\cdot\vec{x}(u_0,v_0).$$

Ex: Consider the sphere w/ local coords $\vec{x}(\theta,\varphi) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$. What is the tangent plane at $(\pi/4, \pi/3)$?

Well, $\vec{x}_\theta = \begin{bmatrix}\cos\varphi\cos\theta\\ \sin\varphi\cos\theta\\ -\sin\theta\end{bmatrix}$ & $\vec{x}_\varphi = \begin{bmatrix}-\sin\varphi\sin\theta\\ \cos\varphi\sin\theta\\ 0\end{bmatrix}$,

so $\vec{n} = \vec{x}_\theta \times \vec{x}_\varphi = \begin{bmatrix}\cos\varphi\sin^2\theta\\ -\sin\varphi\sin^2\theta\\ \cos^2\varphi\sin\theta\cos\theta + \sin^2\varphi\sin\theta\cos\theta\end{bmatrix} = \sin\theta\begin{bmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{bmatrix} = \sin\theta\,\vec{x}(\theta,\varphi)$

So now, when $\theta = \pi/4$ & $\varphi = \pi/3$, we have that $\vec{n}(\pi/4,\pi/3) = \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\\ \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}1/4\\ \sqrt{3}/4\\ 1/4\end{bmatrix}$

And the tangent plane is given by $\vec{n}(\pi/4,\pi/3)\cdot(x,y,z) = \vec{n}(\pi/4,\pi/3)\cdot\vec{x}(\pi/4,\pi/3)$

$$\frac{1}{4}x + \frac{\sqrt{3}}{4}y + \frac{1}{4}z = \sin(\pi/4) = \frac{1}{\sqrt{2}}$$