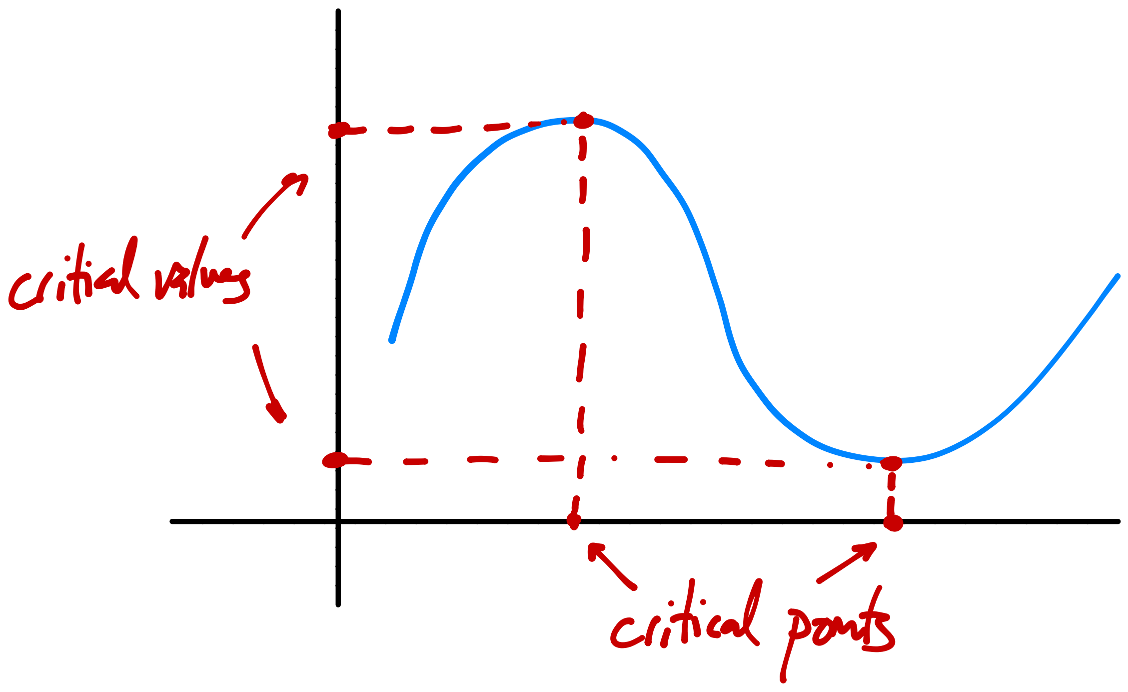
Math 474: Day 18

We saw last time that surfaces locally look like graphs of functions $f:\mathbb{R}^2\to\mathbb{R}$. This is helpful for calculations, but not really so helpful for constructing surfaces. For that, a different connection to functions is useful.

Def: Given a differentiable map $F:U\subseteq\mathbb{R}^n\to\mathbb{R}^m$, a point $p\in U$ is a critical point of F if $dF_p:\mathbb{R}^n\to\mathbb{R}^m$ is not onto (i.e., has rank $<m$). The image $F(p)$ is called a critical value, & points in the range of F which are not critical values are regular values.



Prop: If $F: U\subseteq\mathbb{R}^3\to\mathbb{R}$ is a differentiable function & $\alpha\in F(U)$ is a regular value, then $F^{-1}(\alpha)$ is a regular surface in $\mathbb{R}^3$.

I won't bore you with the proof (which you can find in §2-2 of do Carmo), but it's similar to our previous proof, except it depends on the Implicit Function Thm rather than the Inverse Function Thm.

Ex: The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is a regular surface (note that this includes the sphere when $a=b=c=1$). To check, notice that the ellipsoid is $F^{-1}(0)$ for $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$. Then

$$dF=\begin{bmatrix}\frac{2x}{a^2} & \frac{2y}{b^2} & \frac{2z}{c^2}\end{bmatrix}.$$

This fails to have rank 1 only when $x=y=z=0$, so the only critical value is $F(0,0,0)=-1$. Hence, 0 is a regular value & $F^{-1}(0)$ is a regular surface.

Ex: The hyperboloid $-x^2-y^2+z^2=c$ is usually a regular surface. When can things go wrong?

Well, the hyperboloid is $F^{-1}(c)$ for $F(x,y,z)=-x^2-y^2+z^2$. Now,

$$dF=\begin{bmatrix}-2x & -2y & 2z\end{bmatrix},$$

which is rank 1 except when $x=y=z=0$, so the only critical value is $F(0,0,0)=0$ & indeed we see in the animation that $F^{-1}(0)$ is a cone, which is not a regular surface.

Notice that when $c>0$ the hyperboloid is disconnected (this is usually called the two-sheeted hyperboloid), which is totally fine by the definition of regular surface.

This is a very useful way to construct regular surfaces & to prove that things are regular surfaces.

Now we turn to the problem of developing a theory of functions on surfaces. The motivating question is:

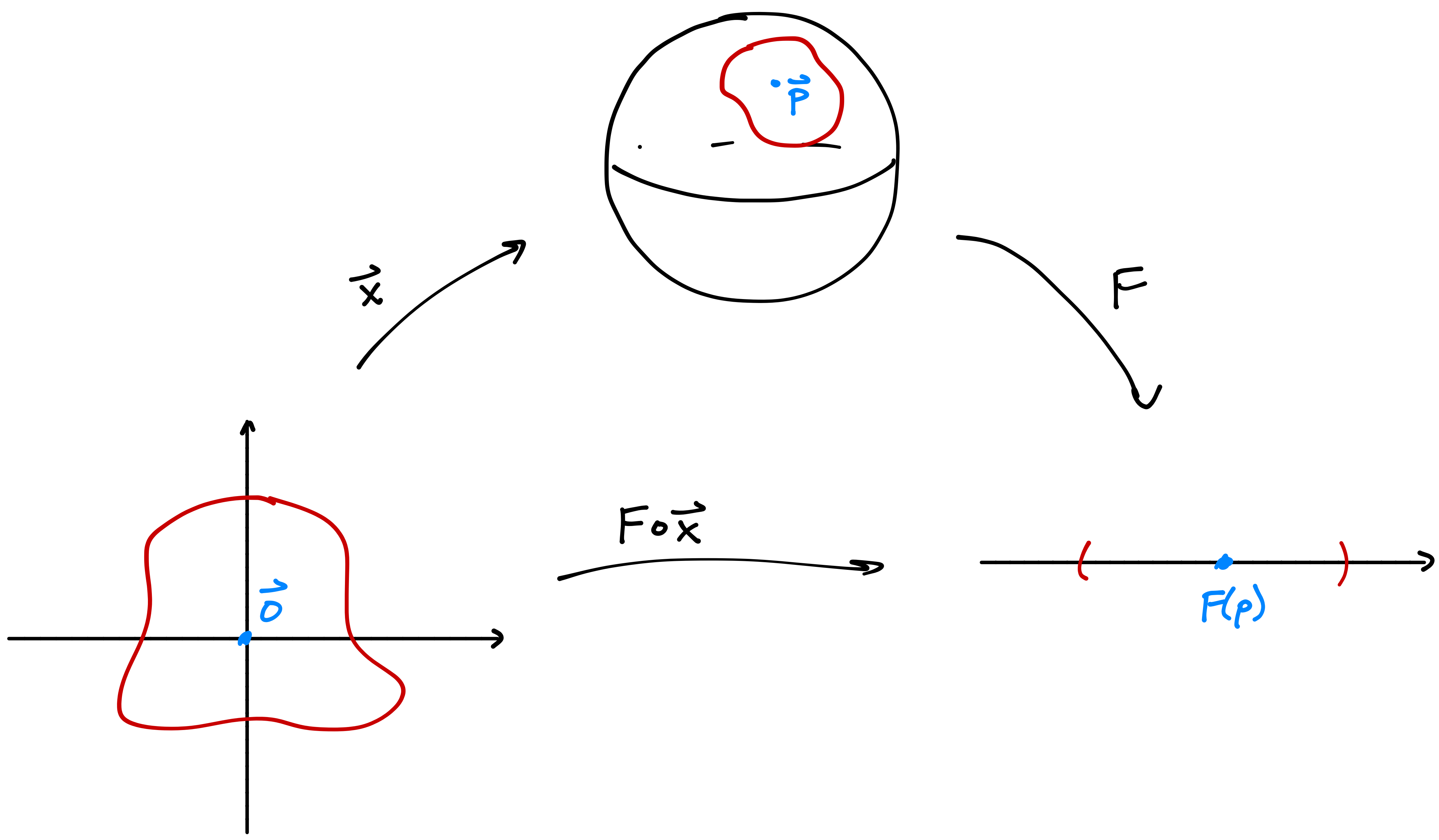
What does it mean to say that a function on a regular surface Σ is "differentiable"?

Or, maybe even better:

Differentiable with respect to what?

The short answer is: with respect to the local coordinates on Σ:

Def: Let $F: V \subseteq \Sigma \to \mathbb{R}$ be a function defined on an open set of the regular surface Σ. Then F is differentiable at $\vec{p} \in V \subseteq \Sigma$ if, for some parametrization $\vec{x}: U \subseteq \mathbb{R}^2 \to \Sigma$ w/ $\vec{x}(\vec{0}) = \vec{p}$, the composition $F\circ\vec{x}: U \to \mathbb{R}$ is differentiable at $\vec{0}$.



In other words, using local coords we've turned F into a function on $\mathbb{R}^2$, where we know perfectly well what it means to be differentiable.

Of course, the thing you should be concerned about is: what if you pick the wrong local coordinates? Could $F\circ\vec{x}$ be diff'ble while $F\circ\vec{y}$ isn't?

As it turns out, the answer is no, this doesn't happen.