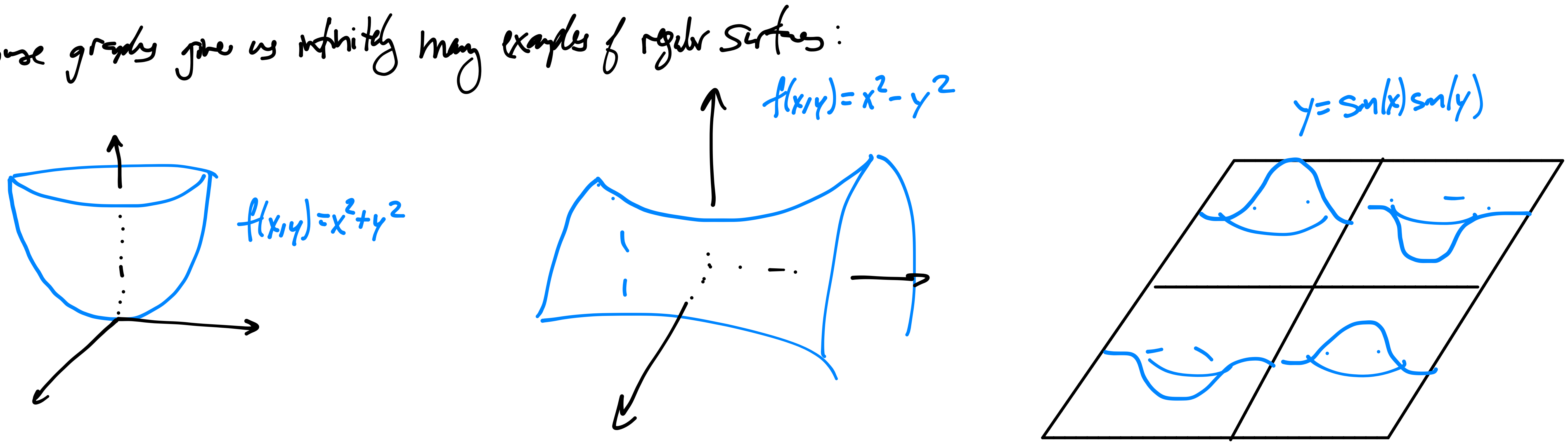
Math 474: Day 17

Prop: If $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, then the graph $\{(x,y,f(x,y)) : (x,y) \in U\}$ is a regular surface.

Proof: Parametrize by $\vec{x}(x,y) = (x,y,f(x,y))$. This is continuous, bijective, & has continuous inverse $\vec{x}^{-1}(x,y,f(x,y)) = (x,y)$. Moreover,

$$d\vec{x}_{(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix}$$ has lin. ind. columns. ∎

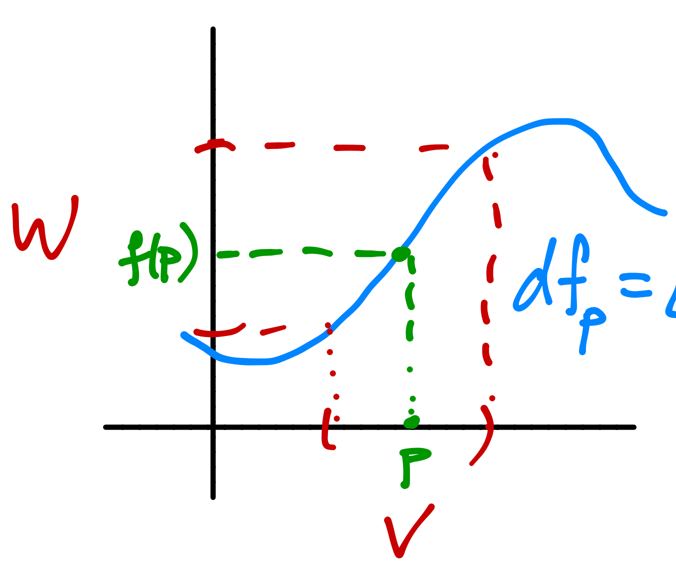
So of course graphs give us infinitely many examples of regular surfaces:



In fact, the slightly amazing thing is that any regular surface is locally a graph of a function.

Inverse Function Thm: Let $F: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map & suppose that at $\vec{p} \in U$ the total derivative $dF_{\vec{p}}: \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism (i.e., nonsingular matrix). Then $\exists$ nbhd V of $\vec{p}$ & W of $F(\vec{p})$ s.t. $F^{-1}: W \to V$ is a well-defined smooth map.

Ex:



$df_p = [f'(p)]$ is an isomorphism iff $f'(p) \neq 0$.

Prop: Let $\Sigma \subset \mathbb{R}^3$ be a regular surface & let $\vec{p} \in \Sigma$. Then $\exists$ nbhd V of $\vec{p}$ so that $V \cap \Sigma$ is the graph of a diff'ble function of the form $z = f(x,y)$ or $y = g(x,z)$ or $x = h(y,z)$.

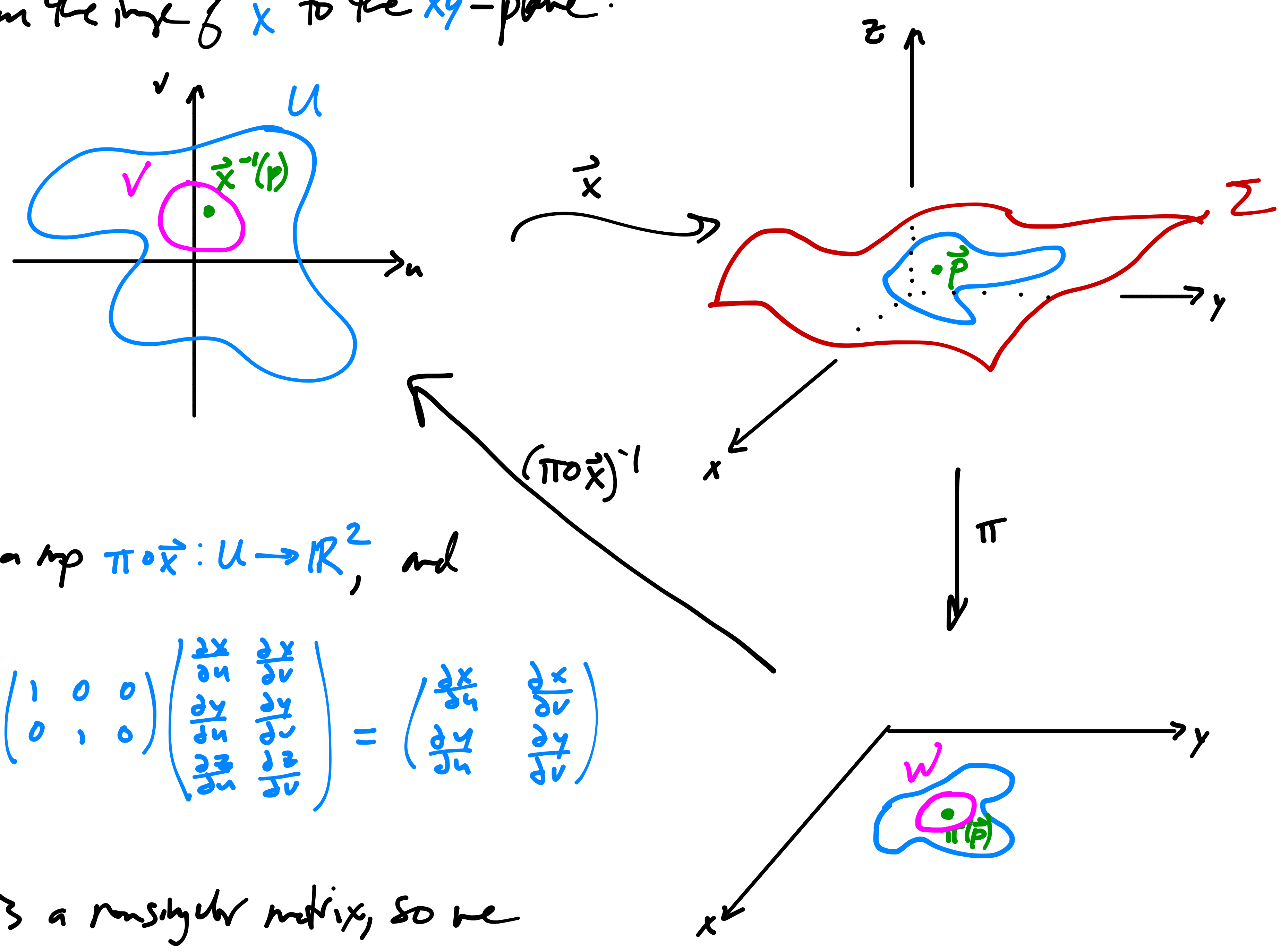
Proof: Let $\vec{p} \in \Sigma$. Since Σ is a regular surface, there is a parametrization $\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ near $\vec{p}$ & we can write

$$\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v)).$$

Now, by ③ we know that $\vec{x}_u \times \vec{x}_v \neq \vec{0}$, so at least one of the coordinates is nonzero. Assume it's the z-coordinate (since we can just modify the following appropriately if it's the x or y), which is:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

Now, we project from the image of $\vec{x}$ to the xy-plane:



By composition, we get a map $\pi\circ\vec{x} : U \to \mathbb{R}^2$, and

$$d(\pi\circ\vec{x}) = d\pi\, d\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

But we know this is a nonsingular matrix, so we can use the Inverse Function Thm to see that $\exists$ a nbhd W of $\pi(\vec{p})$ & a nbhd V of $\vec{x}^{-1}(p)$ s.t. there is a differentiable inverse map $(\pi\circ\vec{x})^{-1} : W \to V$.

Then I claim that, near $\vec{p}$, Σ is the graph of the function $z\circ(\pi\circ\vec{x})^{-1}$, namely $\{(x, y, (z\circ(\pi\circ\vec{x})^{-1})(x,y)) : (x,y)\in W\}$.

Of course, we need to know that points on the graph really are points on the surface: for $(x,y)\in W$, we know $(\vec{x}\circ(\pi\circ\vec{x})^{-1})(x,y)$ is on Σ, since $(\pi\circ\vec{x})^{-1}(x,y)\in U$. But now we expand that point out as

$\left(x((\pi\circ\vec{x})^{-1}(x,y)),\ y((\pi\circ\vec{x})^{-1}(x,y)),\ z((\pi\circ\vec{x})^{-1}(x,y))\right)$. But now $\pi(\vec{x}\circ(\pi\circ\vec{x})^{-1}(x,y)) = (\pi\circ\vec{x})\circ(\pi\circ\vec{x})^{-1}(x,y) = (x,y)$,

so we know $x((\pi\circ\vec{x})^{-1}(x,y)) = x$ & $y((\pi\circ\vec{x})^{-1}(x,y)) = y$, so $(x, y, (z\circ(\pi\circ\vec{x})^{-1})(x,y))\in\Sigma$, as needed. ∎