Math 474: Day 16

Regular Surfaces

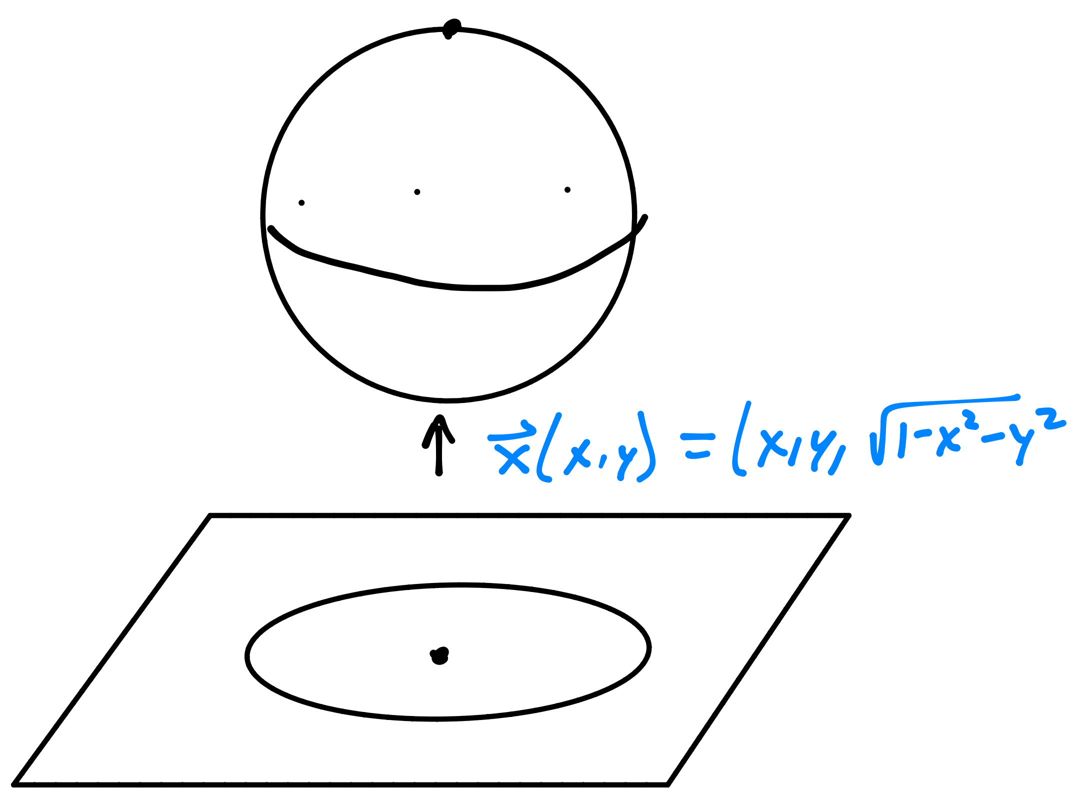
Unfortunately, surfaces turn out to be a lot more complicated than curves. We saw that, although different parametrizations gave the same physical curve, there was a unique "best" parametrization for any curve: the arclength parametrization.

Unfortunately, things don't turn out to be so simple for surfaces, as we see right from the definition of a regular surface:

Def: A subset $\Sigma \subseteq \mathbb{R}^3$ is a **regular surface** if, $\forall\, p \in S$, $\exists$ nbhd V of p & a map $\vec{x}: U \to V \cap \Sigma$, where $U \subseteq \mathbb{R}^2$ is open &:

1. $\vec{x}$ is smooth; i.e., if $\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v))$, then x, y, z have continuous partial derivatives of all orders
2. $\vec{x}$ is a homeomorphism, meaning $\vec{x}$ has a continuous inverse $\vec{x}^{-1}: V \cap \Sigma \to U$ (in particular, this means $\vec{x}$ is bijective)
3. For $q \in U$, $d\vec{x}_q$ is rank 2. Since $d\vec{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} | & | \\ \vec{x}_u & \vec{x}_v \\ | & | \end{pmatrix}$, we know that one of the 2×2 minors has nonzero determinant or, equivalently, $\vec{x}_u \times \vec{x}_v \neq \vec{0}$.

Ex: ① $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ is a regular surface. To see, first define local coords in a nbhd of the north pole $(0,0,1)$:



Clearly differentiable on $U = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$

Now,

$$d\vec{x}_{(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix}$$

which definitely has lin. independent columns, so ③ true

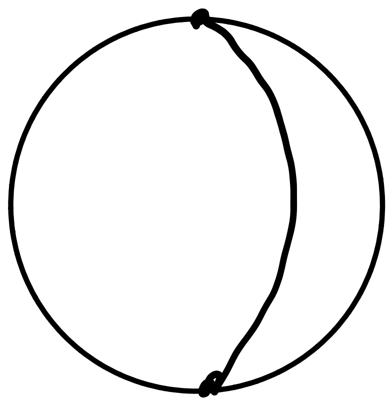
Also, $\vec{x}$ is continuous, injective, & has continuous inverse $\vec{x}^{-1}(x,y,z) = (x,y)$, so $\vec{x}$ satisfies all the local coordinate conditions.

Moreover, the same $\vec{x}$ clearly works for any point in the open northern hemisphere. To cover all the points on the sphere, we could use 6 such coordinates, corresponding to the 6 cardinal directions on the sphere.

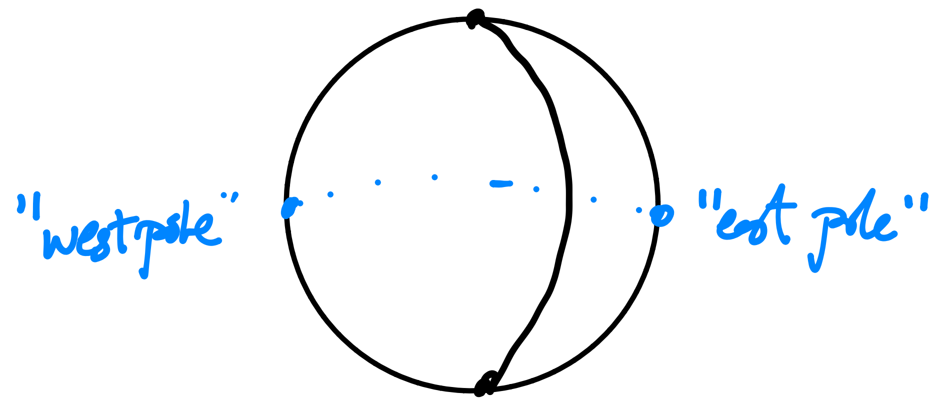
We can do this more efficiently by using (not surprisingly) spherical coordinates

$$(\theta, \varphi) \longmapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\varphi)$$

These coords. are fine everywhere except the north & south poles & the prime meridian



So we could cover S^2 using 2 of these maps:



It turns out you can't do better than this: you always need at least 2 coord. charts on S^2.

This is obviously a pain, so it would be great to prove some theorems that would allow us to show that things are regular surfaces w/o having to explicitly find such charts.

Prop: If $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, then the graph $\{(x, y, f(x,y)) : (x,y) \in U\}$ is a regular surface.