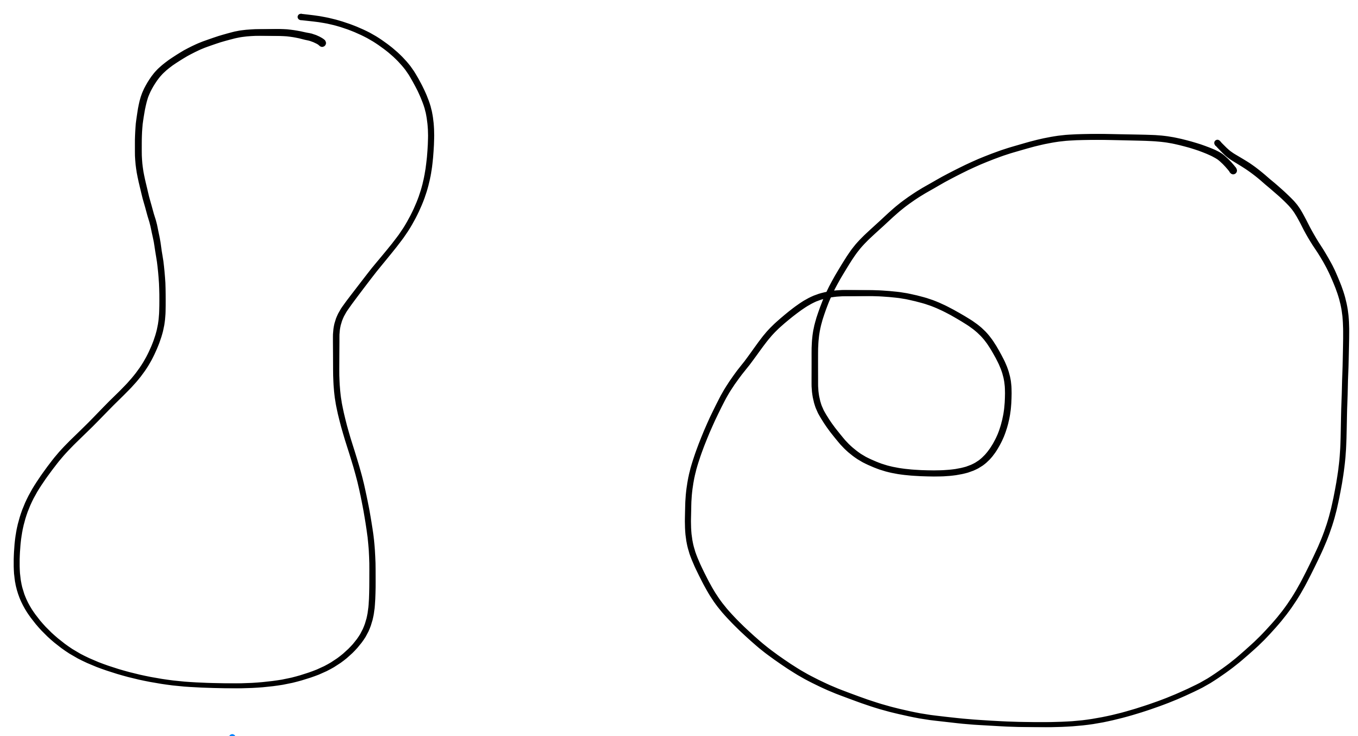
Math 474: Day 15

We'll start moving on to surfaces, but I want to do one last example which kind of motivates the surface stuff.

Remember the rotation index I_α of a plane curve α, which was defined to be the number of turns the tangent indicatrix $T(s)$ goes around the unit circle.

Then visually I counts "loops" in α



$I = \pm 1$ $\qquad\qquad$ $I = \pm 2$

A reasonable question to ask is: to what extent are these loops stable features of α? Can we get rid of them easily, or are they intrinsic to the curve? If they are intrinsic, can we classify closed plane curves by counting their "loop number"?

Now, the key point is to define a suitable notion of equivalence for curves. First of all, recall that

Def: A curve $\alpha(t)$ is regular if $\alpha'(t) \neq 0$ for all t.

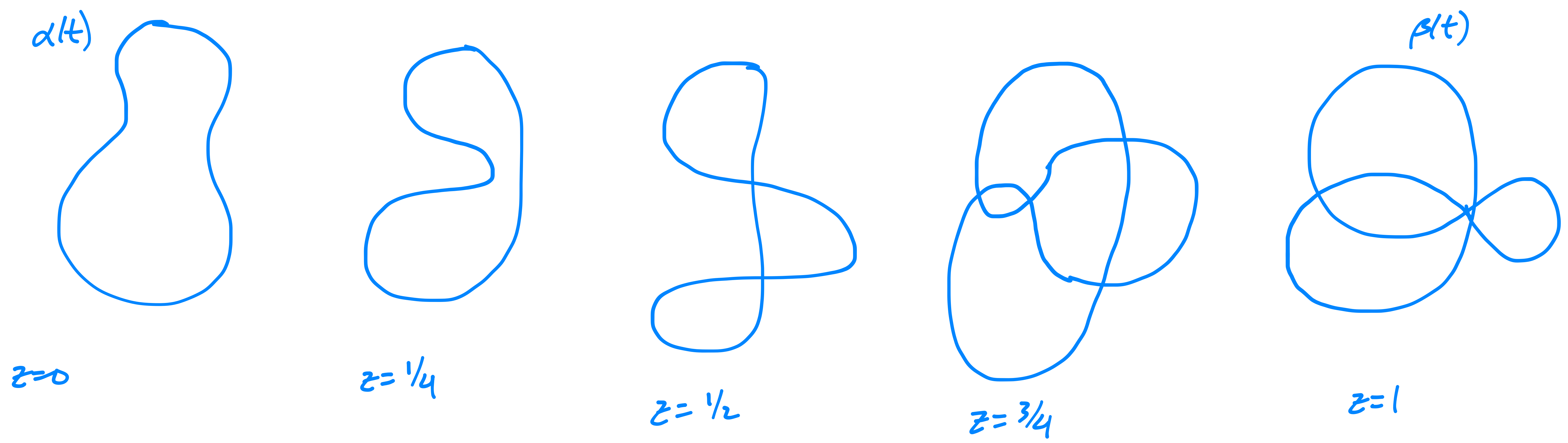
Then we have

Def: A regular homotopy between two closed regular curves $\alpha(t)$ & $\beta(t)$ is a continuously differentiable map

$$H(t,z): [a,b] \times [0,1] \to \mathbb{R}^2$$

so that:
1. $H(t,0) = \alpha(t)$
2. $H(t,1) = \beta(t)$
3. $H(t,z_0)$ is a regular closed smooth curve for all $z_0 \in [0,1]$.

The idea is to think of H as encoding a movie showing α transforming into β:



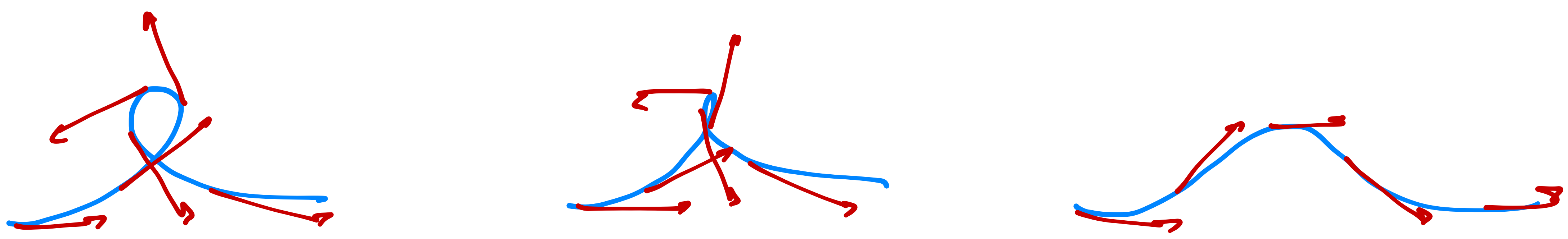
Okay, but the thing you have to be careful about is that not all movies are allowed.

Since $H(t,z)$ is continuously differentiable w.r.t. both t and z, it must be the case that

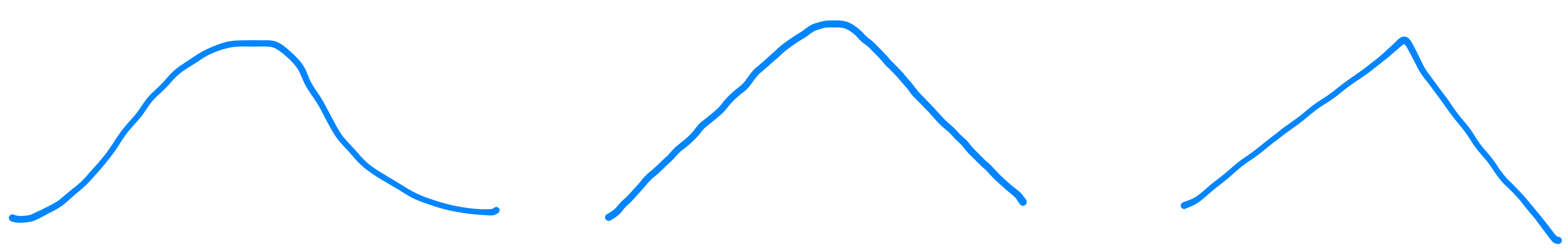
$$\frac{\partial H}{\partial t}(t,z) \text{ is continuous in } z.$$

But of course $\frac{\partial H}{\partial t}(t,z_0)$ is giving the tangent vectors of the regular curve $H(t,z_0)$. So this means the tangent vectors depend continuously on z.

So... can we pull a loop tight?



No! Can we make a corner?

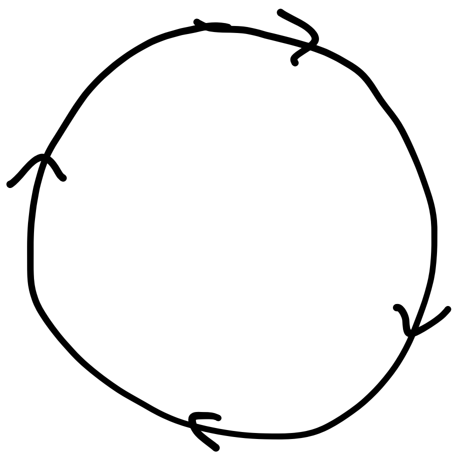


No! In particular...

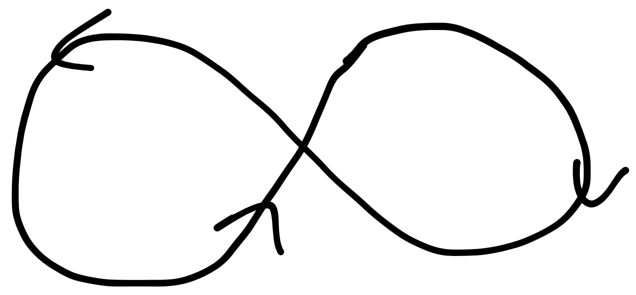
Prop: The rotation index of $H(t,z)$ depends continuously on z. Thus, it is constant.

Proof: The rotation index is defined by the angle swept out by the tangent indicatrix to $H(t,z)$. But since the tangent indicatrix depends continuously on z, this angle does, too. (Where did we implicitly use regularity?) ∎

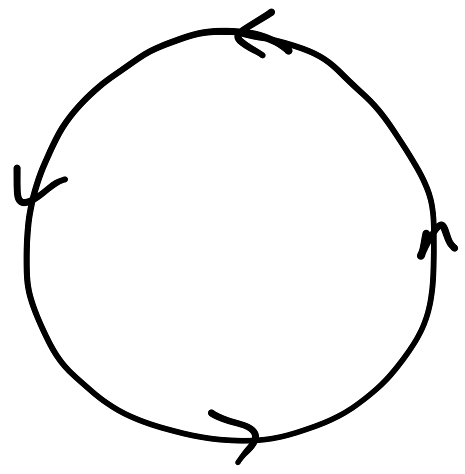
In particular, we now know you **can't** turn a circle inside-out!



$I=-1$



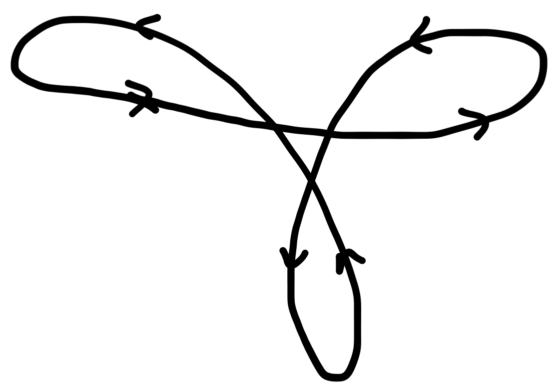
$I=0$



$I=1$

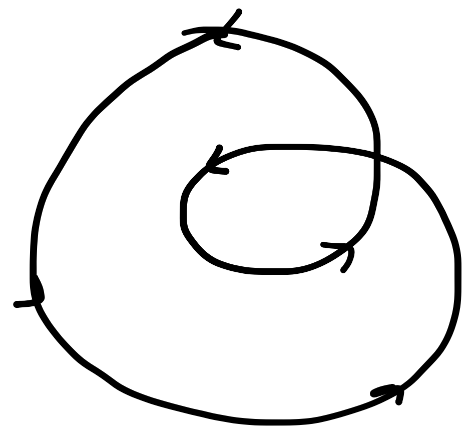
None of these can be regularly homotopic.

Okay, what about the converse? For example, is



$I=2$

regularly homotopic to



$I=2$

?

Yes! In fact...

Whitney-Graustein Thm: All regular closed plane curves w/ the same rotation index are regularly homotopic.