Math 474: Day 14

Wirtinger's Inequality: Let $h:\mathbb{R}\to\mathbb{R}$ be a periodic function of period 2π which is continuously differentiable & satisfying

$$\int_0^{2\pi} h(t)\,dt = 0.$$

Then $\int_0^{2\pi}(h'(t))^2\,dt \geq \int_0^{2\pi}(h(t))^2\,dt$ with equality $\Leftrightarrow h(t) = a\cos t + b\sin t$ for some $a,b\in\mathbb{R}$.

Proof: Since h is periodic & smooth, it is equal to its Fourier series:

$$h(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

where the Fourier coefficient $c_n = \frac{1}{2\pi}\int_0^{2\pi} h(t)\, e^{-int}\,dt$.

Now, we know that $c_0 = \frac{1}{2\pi}\int_0^{2\pi} h(t)\, e^{-i(0)t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} h(t)\,dt = 0$.

But then $h(t) = \sum_{n\neq 0} c_n e^{int}$ & $h'(t) = \sum_{n\neq 0} n c_n e^{int}\,dt$. By Parseval's theorem,

$$\int_0^{2\pi}(h'(t))^2\,dt = 2\pi\sum_{n\neq 0}(nc_n)^2 = 2\pi\sum_{n\neq 0} n^2 c_n^2 \geq 2\pi\sum_{n\neq 0} c_n^2 = \int_0^{2\pi}(h(t))^2\,dt$$

with equality if & only if $c_n = 0$ for $|n|>1$, meaning

$$\begin{aligned} h(t) &= c_1 e^{it} + c_{-1}e^{-it} = c_1 e^{it} + \overline{c_1} e^{-it} = c_1 e^{it} + \overline{(c_1 e^{it})} \\ &= 2\,\mathrm{Re}(c_1 e^{it}) \\ &= \mathrm{Re}(c_1)\cos t - \mathrm{Im}(c_1)\sin t \\ &= a\cos t + b\sin t \end{aligned}$$

for $a = \mathrm{Re}(c_1)$ & $b = -\mathrm{Im}(c_1)$. ▨

Remember, this was all in service of proving:

Isoperimetric Inequality: Suppose α is a simple (i.e., non-self-intersecting) closed plane curve that encloses an area A.

Then $\mathrm{length}(\alpha)^2 \geq 4\pi A$, with equality if & only if α is a circle.

Proof: For any parametrization $\alpha(t) = (x(t), y(t))$, we know that

$$\mathrm{length}(\alpha) = \int_a^b |\alpha'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

The square root is annoying, so let $t = \frac{2\pi}{L}s$, where $s = s(t) = \int_a^t \sqrt{x'(t)^2+y'(t)^2}\,dt$ is the arclength parameter.

Then $\int_0^{2\pi}(x'(t)^2 + y'(t)^2)\,dt = \int_0^{2\pi}(s'(t))^2\,dt = \int_0^{2\pi}\left(\frac{L}{2\pi}\right)^2 dt = \frac{L^2}{2\pi}$.

Now we're trying to show $L^2 \geq 4\pi A$ or, equivalently, $L^2 - 4\pi A \geq 0$.

But, by the corollary, $A = -\int_0^{2\pi} y(t)x'(t)\,dt$, so

$$L^2 - 4\pi A = 2\pi\int_0^{2\pi}(x'(t)^2 + y'(t)^2)\,dt + 4\pi\int_0^{2\pi} y(t)\,x'(t)\,dt = 2\pi\int_0^{2\pi}\left(x'(t)^2 + (y'(t))^2 + 2y(t)\,x'(t)\right)dt$$

$$= 2\pi\int_0^{2\pi}(x'(t) + y(t))^2\,dt + 2\pi\int_0^{2\pi}\left(y'(t)^2 - y^2\right)dt$$

The first is obviously ≥ 0. The second will vanish by Wirtinger's Inequality provided we have $\int_0^{2\pi} y(t)\,dt = 0$. But this is easy to ensure: just choose coordinates so that the x-axis passes through the center of mass of the curve.

So this takes care of the $\geq$ part of the theorem.

Now, if $L^2 = 4\pi A$, then both integrals in the above must vanish. The first has non-negative integrand, so the integrand itself must vanish identically, meaning $x'(t) = -y(t)\ \forall t$.

As for the second integral, vanishing implies $\int_0^{2\pi}(y'(t))^2\,dt = \int_0^{2\pi} y(t)^2\,dt$. By the equality part of Wirtinger's Inequality, this means that

$$y(t) = a\cos(t) + b\sin(t)$$

Since $x'(t) = -y(t)$, we see that $x(t) = -a\sin t + b\cos t + c$, so our curve is

$$\alpha(t) = (-a\sin t + b\cos t + c,\ a\cos t + b\sin t),$$

which is just a funky parametrization of the circle of radius $\sqrt{a^2+b^2}$ centered at $(c, 0)$. ∎