Math 474: Day 13

Isoperimetric Inequality: Suppose α is a simple (i.e., non-self-intersecting) closed plane curve that encloses an area A.

Then $\text{length}(\alpha)^2 \geq 4\pi A$, with equality if & only if α is a circle.

First off, let's pause to appreciate what this says: if you have 5000 ft of fence to enclose a livestock pen, you will give your cows maximum room to move around by making a circle of radius $\frac{5000}{2\pi}$.... you can't possibly win by making some crazy shape.

Will need to have some preliminary facts:

Green's Thm: Let $R \subseteq \mathbb{R}^2$ be a region, let ∂R be its boundary curve, oriented counterclockwise, & suppose P & Q are continuously differentiable functions on R. Then

$$\int_{\partial R} \left(P(u,v)\,du + Q(u,v)\,dv\right) = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv$$

Corollary: If $\alpha(t) = (x(t), y(t))$ is a closed curve, then the area enclosed by α is

$$A = \int_\alpha x(t)\,y'(t)\,dt = -\int_\alpha y(t)\,x'(t)\,dt$$

Pf: By defn, if R is the enclosed region,

$$A = \iint_R 1\,dx\,dy \overset{\text{Green's}}{=} \int_\alpha x\,dy = \int_\alpha x\,y'(t)\,dt \qquad \text{by letting } Q(x,y) = x \;\&\; P(x,y) = 0$$

$$\text{or} \quad = -\int_\alpha y\,dx = -\int_\alpha y(t)\,x'(t)\,dt \qquad \text{by letting } Q(x,y) = 0 \;\&\; P(x,y) = y$$

$\blacksquare$

Wirtinger's Inequality: Let $h: \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π which is continuously differentiable & satisfying

$$\int_0^{2\pi} h(t)\,dt = 0.$$

Then $\int_0^{2\pi} (h'(t))^2\,dt \geq \int_0^{2\pi} (h(t))^2\,dt$ with equality $\Leftrightarrow h(t) = a\cos t + b\sin t$ for some $a, b \in \mathbb{R}$.

Proof: Since h is periodic & smooth, it is equal to its Fourier series:

$$h(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

where the Fourier coefficient $c_n = \frac{1}{2\pi}\int_0^{2\pi} h(t)\,e^{-int}\,dt$.