Math 474: Day 12

Fenchel's Thm: The total curvature of any closed curve is at least 2π, with equality if & only if the curve is a convex plane curve.

Proof: Let $T(s)$ be the tangent indicatrix of the curve α. Then by earlier considerations $\text{length}(T) = \int_\alpha \kappa(s)\,ds$.

If α is a convex plane curve, then T is clearly a great circle, which has length 2π.

Now suppose α is a space curve & $T(s)$ is its tangent indicatrix, which is a curve on the unit sphere.

Then the total curvature of α is $\int_0^\ell \kappa(s)\,ds = \int_0^\ell |T'(s)|\,ds = \text{length}(T)$, so we're done if we can show $\text{length}(T) \geq 2\pi$ & that $\text{length}(T) = 2\pi$ implies α is planar & convex.

By the spherical Crofton formula, $\text{length}(T) = \frac{1}{4}\int_{\vec{p}\in S^2} I_T(\vec{p}^{\perp})\,d\vec{p} = \pi(\text{avg. \# of intersections of } T \text{ w/ a great circle})$.

In other words we need to show that the avg. # of intersections of T w/ a great circle is ≥ 2.

Generically, if T intersects a great circle at all, it intersects that great circle at least twice, so we just need to show

that T intersects (almost) any great circle. We state & prove this as a separate sequence of lemmas:

Lemma 1: A spherical curve T is the tangent indicatrix of an arclength parametrized closed curve if & only if $\int_0^\ell T(s)\,ds = 0$.

Pf: If α is closed & $T(s) = \alpha'(s)$, then $\int_0^\ell T(s) = \int_0^\ell \alpha'(s)\,ds = \alpha(\ell) - \alpha(0) = 0$.

OTOH, if $\int_0^\ell T(s)\,ds = 0$, then $\alpha(s) = \alpha(0) + \int_0^s T(u)\,du$ is closed, since $\alpha(\ell) = \alpha(0) + \int_0^\ell T(s)\,ds = \alpha(0) + \vec{0} = \alpha(0)$.

Lemma 2: If $T(s) = \alpha'(s)$ is the tangent indicatrix of a closed curve $\alpha(s)$, then T intersects any great circle on S^2.

Proof: Consider $T(s)\cdot\vec{p}$ for $\vec{p}\in S^2$. If we average over all points on T, we get

$$\int_0^\ell T(s)\cdot\vec{p}\,ds = \vec{p}\cdot\int_0^\ell T(s)\,ds = 0,$$

so the average value of $T(s)\cdot\vec{p}$ is 0. But that means $T(s)$ must intersect $\vec{p}^{\perp}$ (otherwise, $T(s)\cdot\vec{p}$ would be either always positive or always negative). ∎

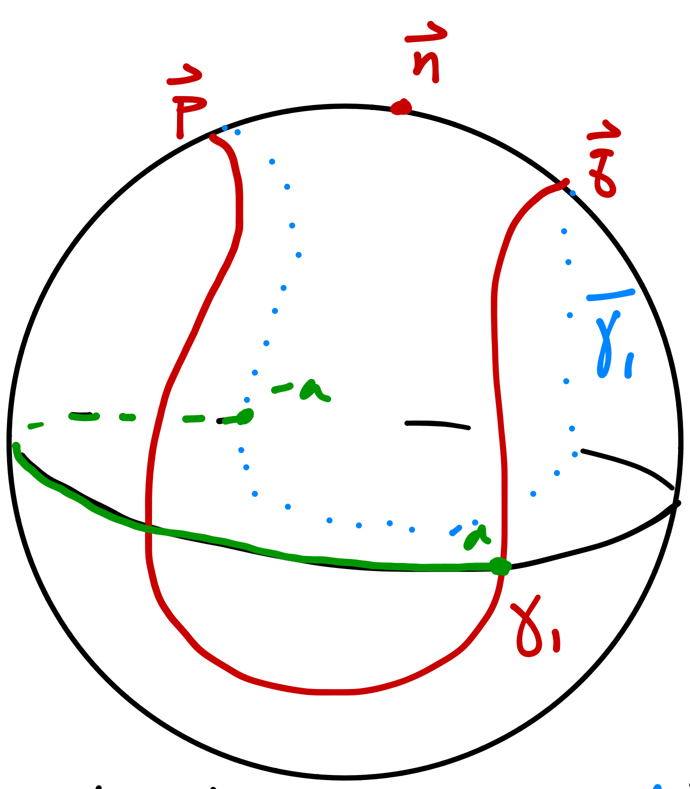
So with these 2 lemmas in place, we've shown that $\text{length}(T) \geq 2\pi$.

Now, the equality case is a bit tricky: we need to show that if $\int \kappa(s)\,ds = \text{length}(T) = 2\pi$, then α is planar & convex. The result will follow if we can show that T is a great circle, which will follow from

Lemma 3: If γ is a spherical curve with $\text{length}(\gamma) \leq 2\pi$, then γ lies in a closed hemisphere.

This suffices to prove the claim, b/c the only way to lie in a hemisphere & to have $\int T\cdot\vec{p}\,ds = 0$ where $\vec{p}$ is the north pole is to be the equator.

Proof of Lemma 3: Suppose $\text{length}(\gamma) \le 2\pi$. Break γ up into 2 arcs γ_1 & γ_2 of the same length & let $\{\vec{p}, \vec{q}\} := \gamma_1 \cap \gamma_2$. Let $\vec{n}$ be the midpoint of the shortest great circle arc connecting $\vec{p}$ & $\vec{q}$.



(If γ is already a great circle, there's no unique choice of $\vec{n}$, so assume γ is not a great circle)

Now, the goal is to show that γ_1 (and hence, by symmetry, γ_2) can't enter the southern hemisphere.

Suppose it did & let $\overline{\gamma}_1$ be the reflection of γ_1 through $\vec{n}$.

Then $\gamma_1 \cup \overline{\gamma}_1$ contains the antipodal points a & $-a$ & is closed, so $\text{length}(\gamma_1 \cup \overline{\gamma}_1) > 2\pi$ since γ not a great circle.

But $\text{length}(\gamma_1 \cup \overline{\gamma}_1) = 2\,\text{length}(\gamma_1) = \text{length}(\gamma)$, so we have a contradiction & γ_1 can't enter the southern hemisphere. ∎

A related but harder theorem (proved by Milnor in his undergraduate senior thesis!)

Fáry-Milnor Thm: If a simple closed curve in space is knotted, then its total curvature is $> 4\pi$.