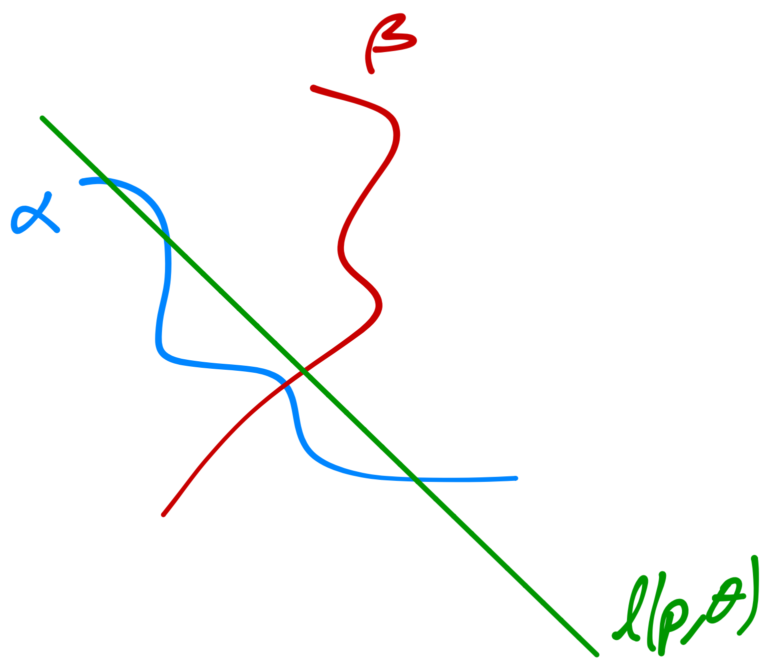
Math 474: Day 11

Last time we showed that Crofton's formula holds for the line segment connecting $(-a,0)$ & $(0,a)$ and that it is invariant under rigid motions of the plane. Since any line segment is congruent to one of the standard line segments, this proves Crofton's formula for any line segment.

Now we just need one more idea: for curves α & β,

$$\int_{\mathcal{L}} I_{\alpha\cup\beta}(p,\theta)\,dp\,d\theta = \int_{\mathcal{L}} I_{\alpha}(p,\theta)\,dp\,d\theta + \int_{\mathcal{L}} I_{\beta}(p,\theta)\,dp\,d\theta$$

since # of intersections w/ $\alpha\cup\beta$ = (# intersections w/ α) + (# intersections w/ β)



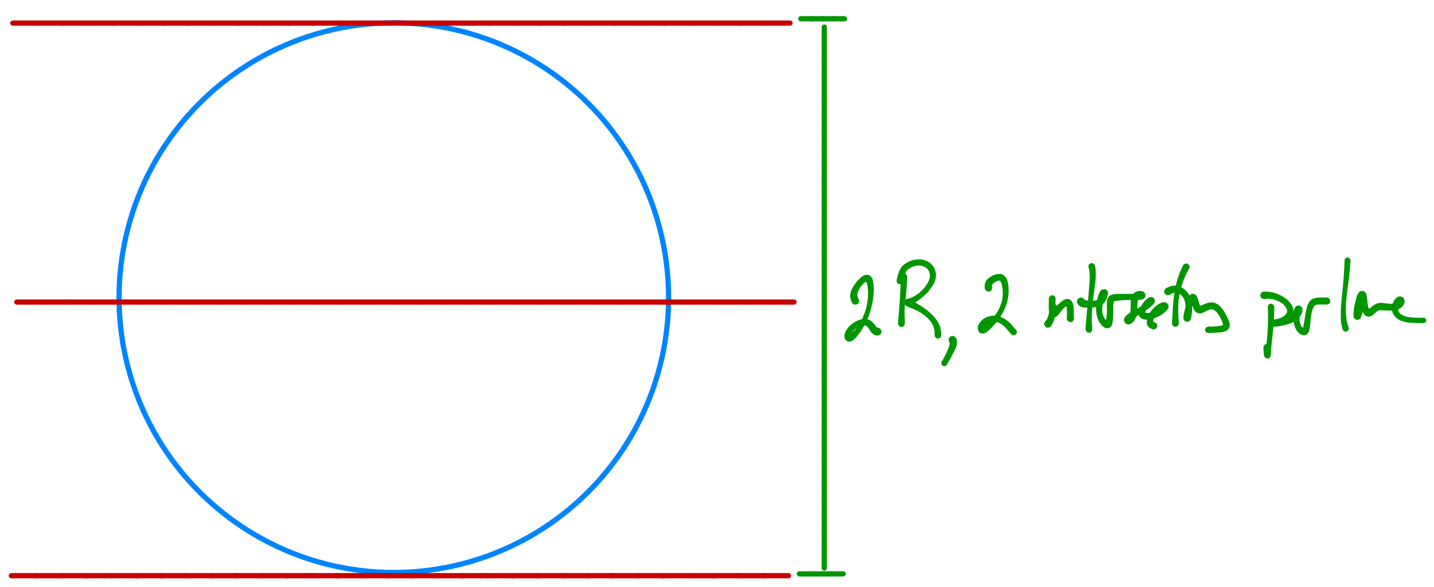
Therefore, for any polygon $\mathbb{P}$ we have that $\int_{\mathcal{L}} I_{\mathbb{P}}(p,\theta)\,dp\,d\theta = 4\,\text{length}(\mathbb{P})$ since $\mathbb{P}$ is a union of line segments.

But now take a sequence $(\mathbb{P}_1, \mathbb{P}_2, \dots)$ converging to α, so that $\text{length}(\mathbb{P}_i) \to \text{length}(\alpha)$. Then we have

$$\int_{\mathcal{L}} I_{\mathbb{P}_i}(p,\theta)\,dp\,d\theta \longrightarrow \int_{\mathcal{L}} I_{\alpha}(p,\theta)\,dp\,d\theta$$

which proves the theorem for arbitrary smooth α.

Ex: Consider the circle of radius R.

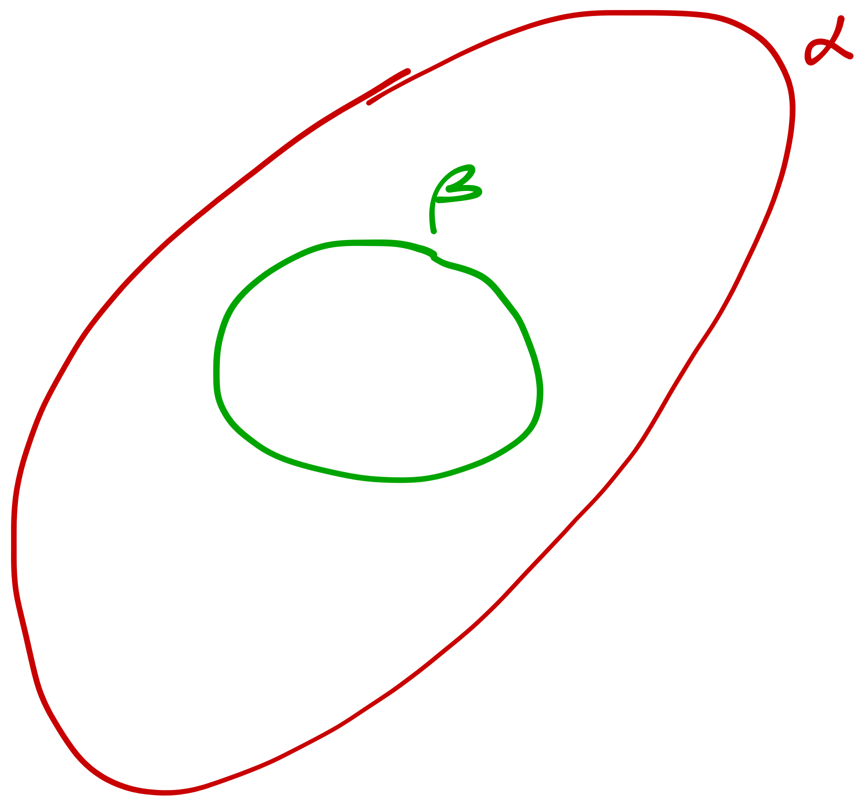


$$\int_{\mathcal{L}} I_{\alpha}(p,\theta)\,dp\,d\theta = \int_{\theta=0}^{2\pi}\left(\int_{-R}^{R} 2\,dp\right)d\theta = \int_{\theta=0}^{2\pi} 4R\,d\theta$$

$$= 8\pi R = 4(\underbrace{2\pi R}_{\text{length}(\alpha)})$$

A neat application:

Suppose α & β are convex curves & β is inside α.



Prop: The probability that a random line through the region determined by α intersects β is $\text{length}(\beta)/\text{length}(\alpha)$

Proof: We know that a line intersecting a convex curve intersects it twice (almost surely), and that any line thru β must intersect α, so

$$P(\ell \text{ intersects } \beta \text{ given that } \ell \text{ intersects } \alpha) = \frac{\text{volume}(\text{lines through } \beta)}{\text{volume}(\text{lines through } \alpha)} \overset{\text{Crofton}}{=} \frac{4\,\text{length}(\beta)}{4\,\text{length}(\alpha)} = \frac{\text{length}(\beta)}{\text{length}(\alpha)}. \qquad \blacksquare$$

(this is sometimes called the Egg Yolk lemma b/c it gives prob. of puncturing yolk w/ needle randomly inserted into an egg)

Corollary 1: $\text{length}(\beta) < \text{length}(\alpha)$.

(Which seems obvious, but is not so easy to prove in other ways)

Corollary 2: If α is any (possibly nonconvex) curve enclosing a convex curve β, then $\text{length}(\alpha) > \text{length}(\beta)$.

$\blacksquare$

Proof: Any line thru β intersects β twice & α at least twice.

A similar formula holds on the sphere:

Thm (Spherical Crofton formula): If α is a curve on the unit sphere, then

$$\text{length}(\alpha) = \frac{1}{4}\int_{S^2} I_\alpha(\vec{p}^{\perp})\, d\vec{p}$$

where $I_\alpha(\vec{p}^{\perp}) = \#$ of intersections of α w/ the great circle perpendicular to $\vec{p} \in S^2$.

We won't prove this, but the point is that it's the same theorem, since great circles on the sphere play the same role as lines in the plane.

Fenchel's Thm: The total curvature of any closed curve is at least 2π, with equality if & only if the curve is a convex plane curve.