Math 474: Day 10

Thm (Crofton's Formula): If α is a curve in $\mathbb{R}^2$, then

$$\text{Length}(\alpha) = 4\iint I_\alpha(p,\theta)\,dp\,d\theta$$

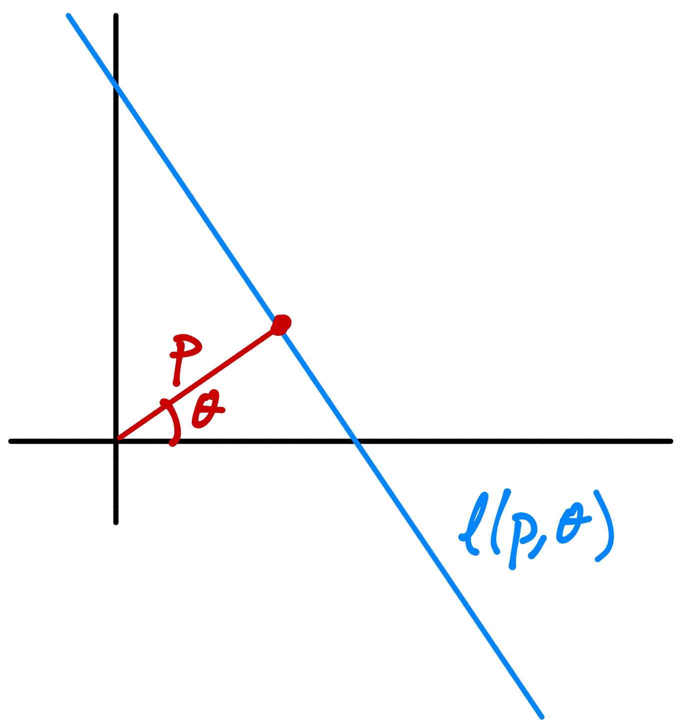
where $I_\alpha(p,\theta)$ is the number of intersections of α w/ the line $\ell(p,\theta)$.

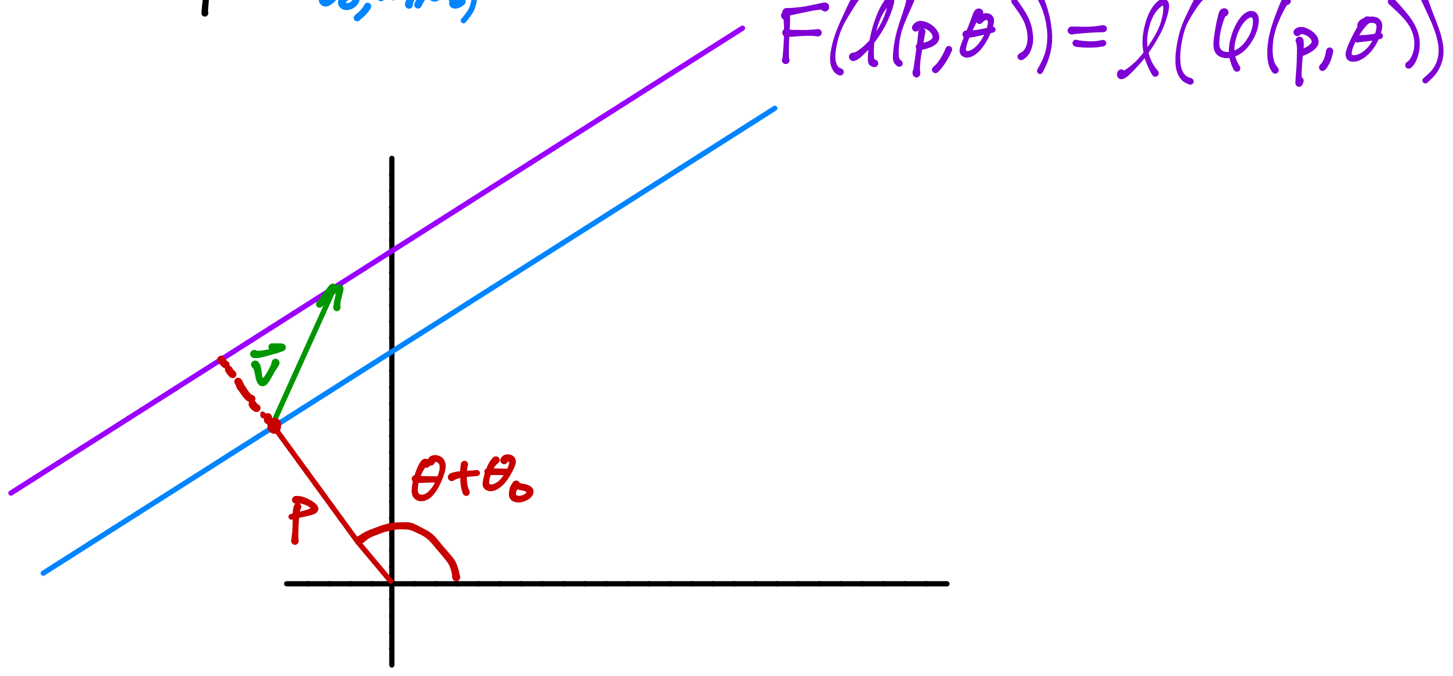
The basic idea is to: ① Prove this for "nice" line segments

② Show that the answer is unchanged by rigid motions, so can extrapolate from line segments to polygons

③ Approximate curves by polygons.

So now, let $F_{\theta_0,(v_1,v_2)}$ be a rigid motion. This induces a map $\varphi_{\theta_0,(v_1,v_2)}$ from $\mathcal{L}$ to $\mathcal{L}$:





And we see that $\varphi_{\theta_0,\vec{v}}(p,\theta) = \left(p + \langle \vec{v}, (\cos(\theta+\theta_0), \sin(\theta+\theta_0))\rangle,\ \theta_0+\theta\right)$

Now, I claim that, for any rigid motion F,

$$\int_{\mathcal{L}} I_{F(\alpha)}(p,\theta)\,dp\,d\theta = \int_{\mathcal{L}} I_\alpha(p,\theta)\,dp\,d\theta.$$

To see this, note that $I_{F(\alpha)}(p,\theta) = \#$ of intersections of $F(\alpha)$ w/ $\ell(p,\theta)$

$= \#$ of intersections of $F(\alpha)$ w/ $F(F^{-1}(\ell(p,\theta))) = F(\ell(\psi(p,\theta)))$ for $\psi = \varphi^{-1}$

$= \#$ of intersections of α w/ $\ell(\psi(p,\theta))$

$= I_\alpha(\psi(p,\theta))$.

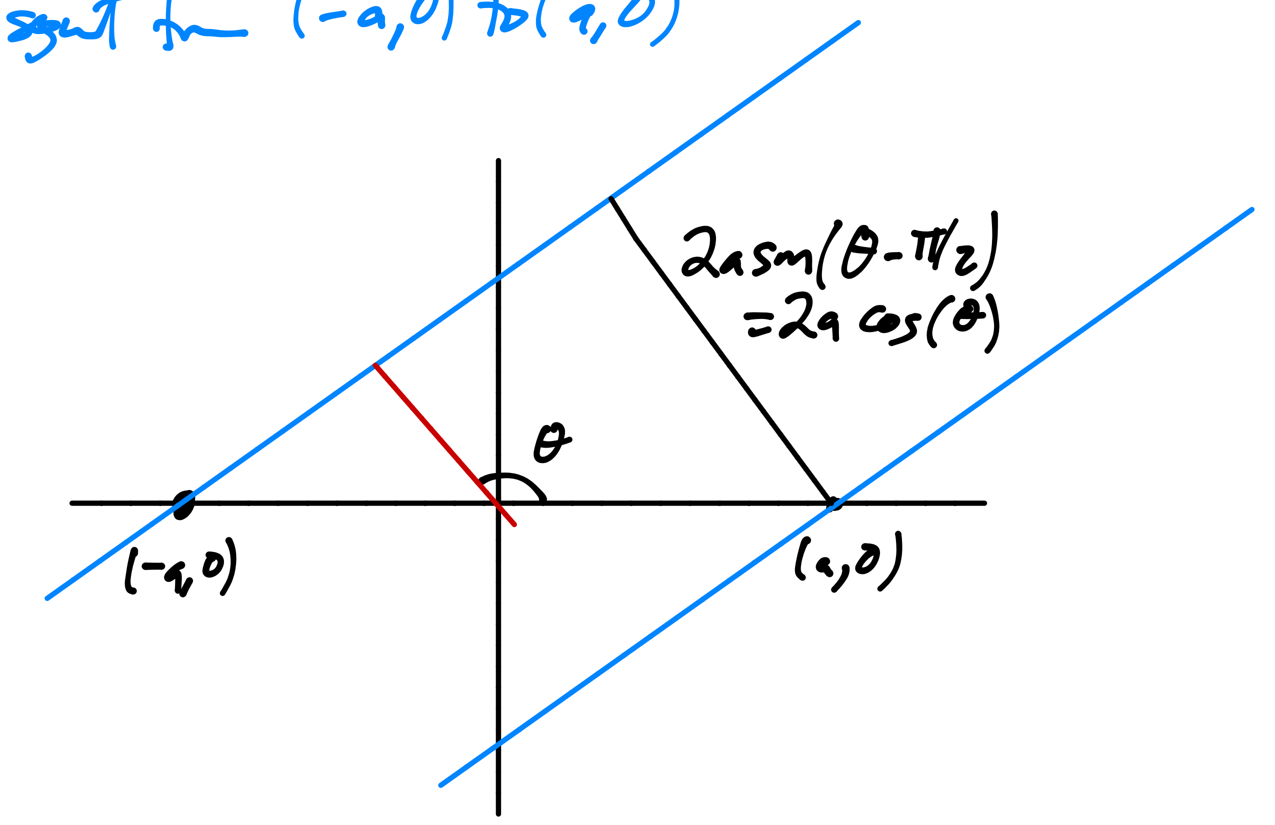
So then $\int_{\mathcal{L}} I_{F(\alpha)}(p,\theta)\,dp\,d\theta = \int_{\mathcal{L}} I_\alpha(\psi(p,\theta))\,dp\,d\theta$.

OTOH, by change-of-variables, $\int_{\mathcal{L}} I_\alpha(p,\theta)\,dp\,d\theta = \int_{\mathcal{L}} I_\alpha(\psi(p,\theta))\,|d\psi|\,dp\,d\theta$, so we're done if $|d\psi| = 1$.

But this is true!

$$d\Psi = \begin{bmatrix} 1 & \frac{\partial}{\partial\theta}\langle \vec{v}, (\cos(\theta+\theta_0), \sin(\theta+\theta_0))\rangle \\ 0 & 1 \end{bmatrix} \text{ so } |d\Psi| = 1.$$

Ex: Consider α = line segment from $(-a,0)$ to $(a,0)$



So $\int_L I_\alpha(p,\theta)\,dp\,d\theta = \int_\theta \left| \int_{-a\cos\theta}^{a\cos\theta} 1\,dp \right| d\theta = \int_\theta 2a|\cos\theta|\,d\theta = 4a\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta = 8a = 4\cdot$(length of line segment)