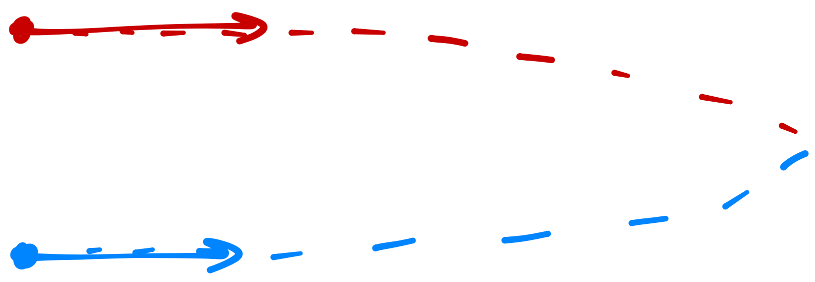
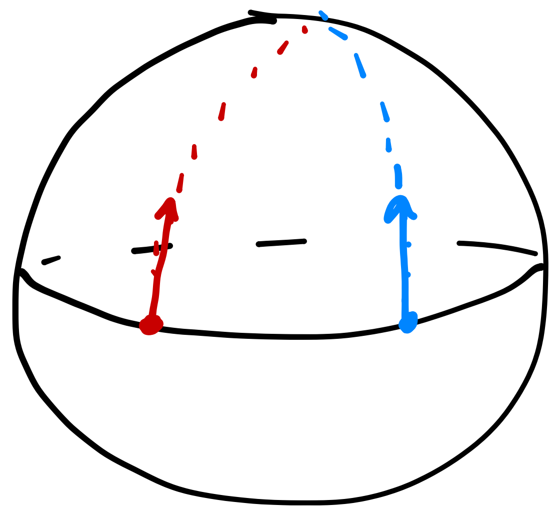
Math 474: Day 1

Imagine 2 particles moving in space along parallel trajectories



Over time, they start to converge... there must be some FORCE pulling them together.

But there's another way to conceptualize this:



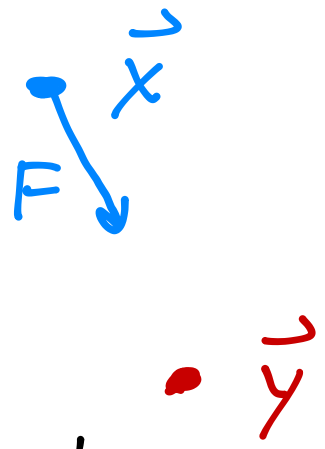
An object moving on a sphere will tend to move along a great circle arc...

So two objects that are initially moving in parallel will eventually meet at the pole!

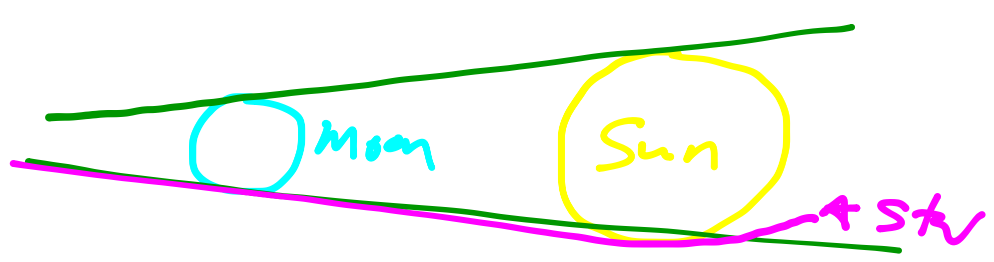
This is the basic idea of Einstein's general relativity: gravity can be interpreted as curving space.

You might ask why bother... after all, we know from Newton that the force of gravity b/w masses at $\vec{x}$ & $\vec{y}$ is

$$F = \frac{\vec{y}-\vec{x}}{|\vec{y}-\vec{x}|^3} \, \mathrm{Mass}(\vec{x})\,\mathrm{Mass}(\vec{y})$$



Of course, the problem is light. Photons are massless, but gravity nonetheless acts on light



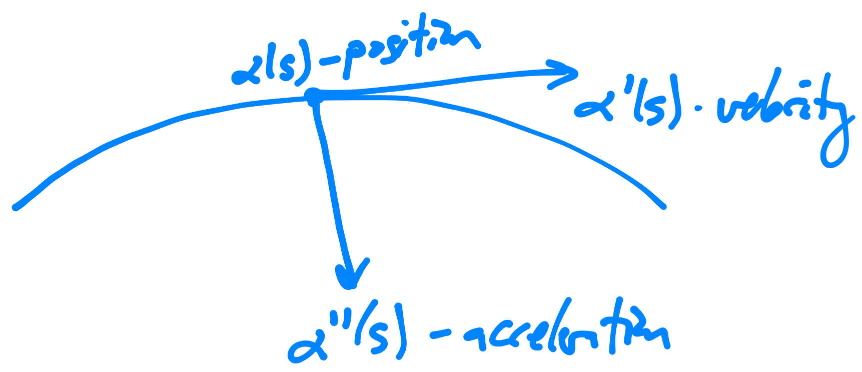
During eclipses we can observe stars which are behind the sun b/c their light is deflected by the sun.

So the idea is that light moves along the shortest path, but the space is curved, so that path isn't a straight line

"Definition": A geodesic in a curved space is a (locally) shortest path b/w 2 points.

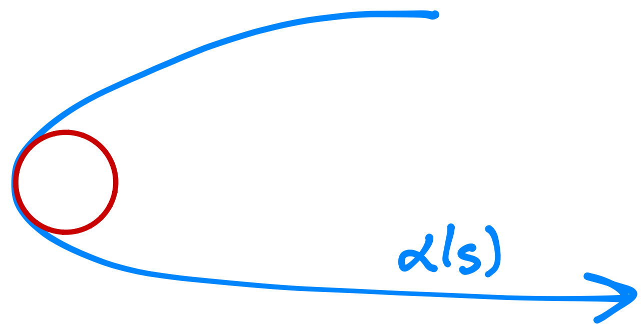
We want to understand such things, which we start doing by understanding parametrized curves:

Def: A parametrized curve $\alpha(s)$ is a function from $\mathbb{R}$ to $\mathbb{R}^3$ (or $\mathbb{R}^2$, or $\mathbb{R}^n$).

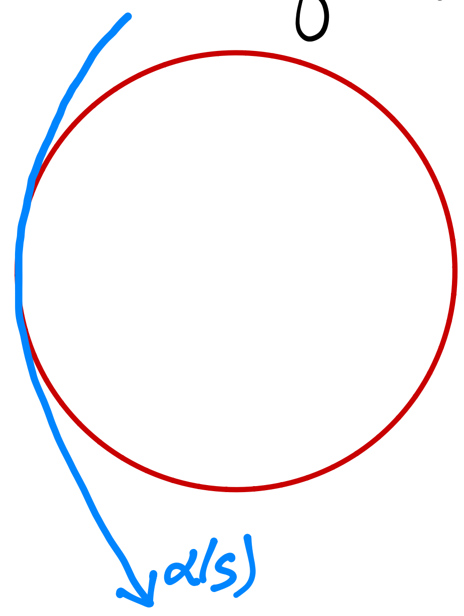


We can understand the local theory of curves by understanding the parametrization & its derivatives.

In particular, the curvature of a curve is a measure of bending



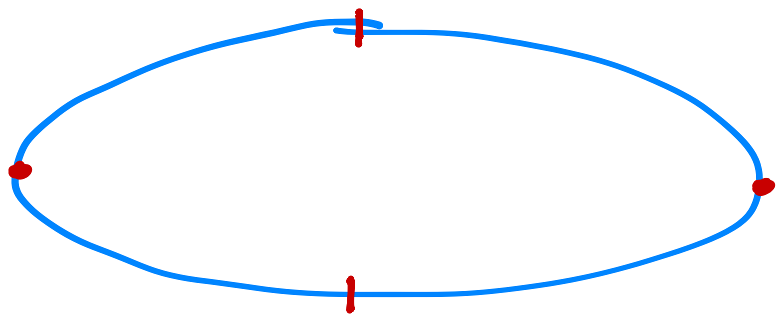
Large curvature - small radius of tangent circle

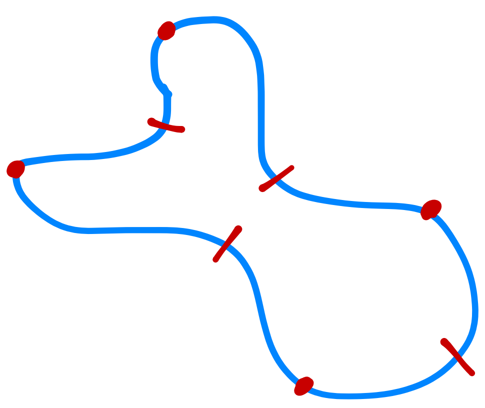


Small curvature - large radius of tangent circle.

There are some really amazing global facts we'll also learn. For example: if $\kappa(s)$ is the curvature of a curve $\alpha(s)$, the critical points of κ are called vertices of the curve.

Four Vertex Thm: Any simple closed plane curve has at least four vertices

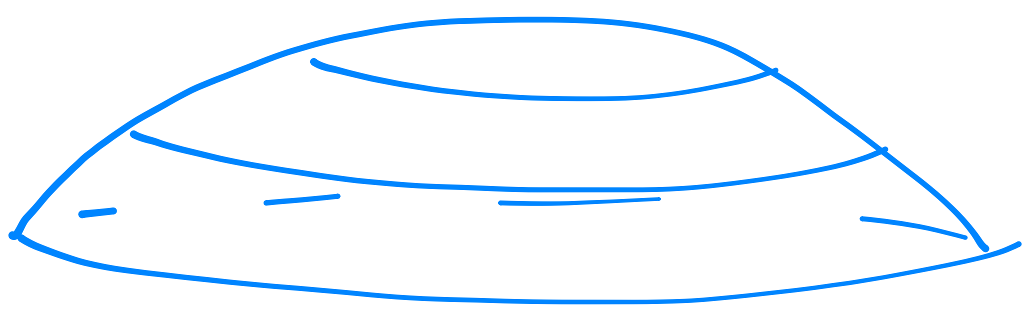




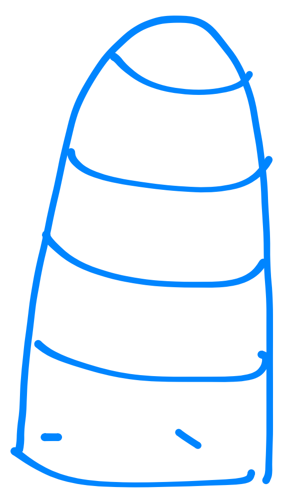
Thm: Any simple closed plane curve has total curvature $\int \kappa(s)\, ds = 2\pi$.

Next, we turn to surfaces, which are sets in $\mathbb{R}^3$ which can locally be written as the image of a map $f: \mathbb{R}^2 \to \mathbb{R}^3$.

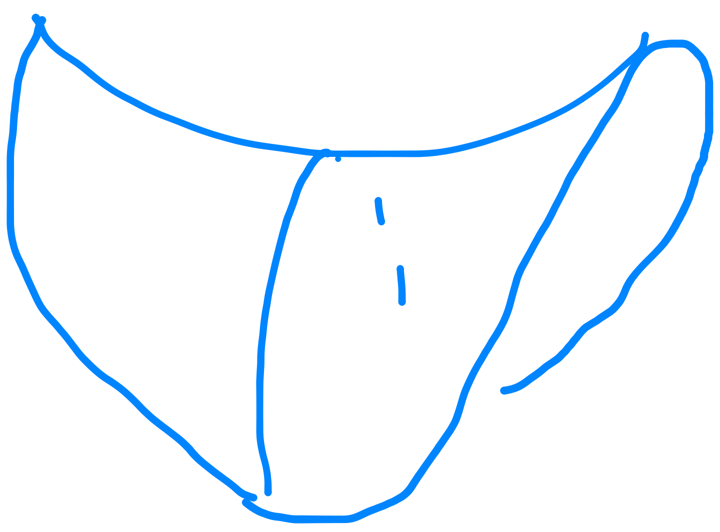
There's also a notion of curvature for surfaces:



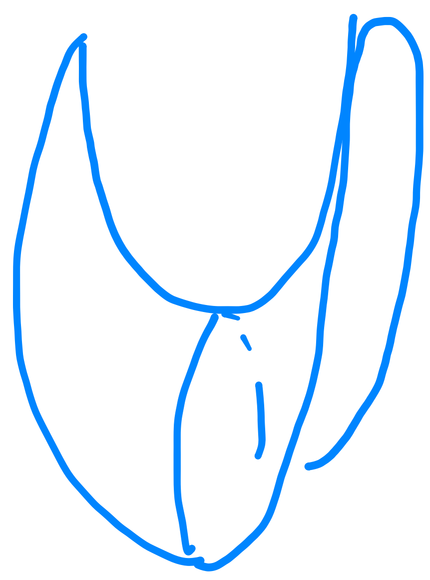
Positive curvature



More positive curvature



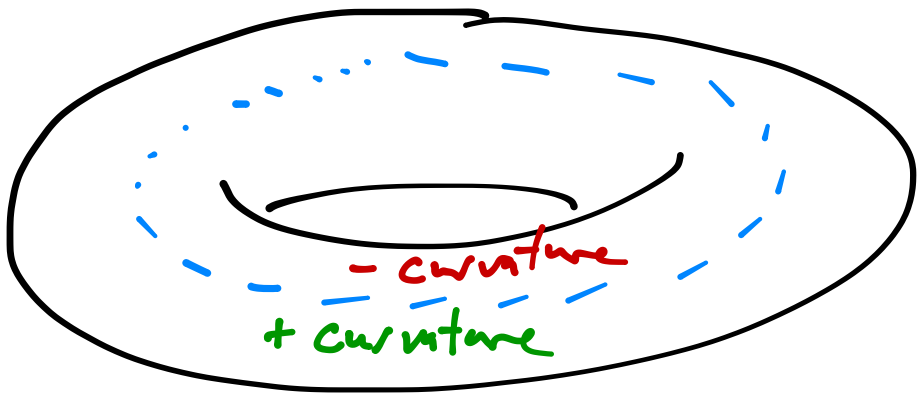
Negative curvature

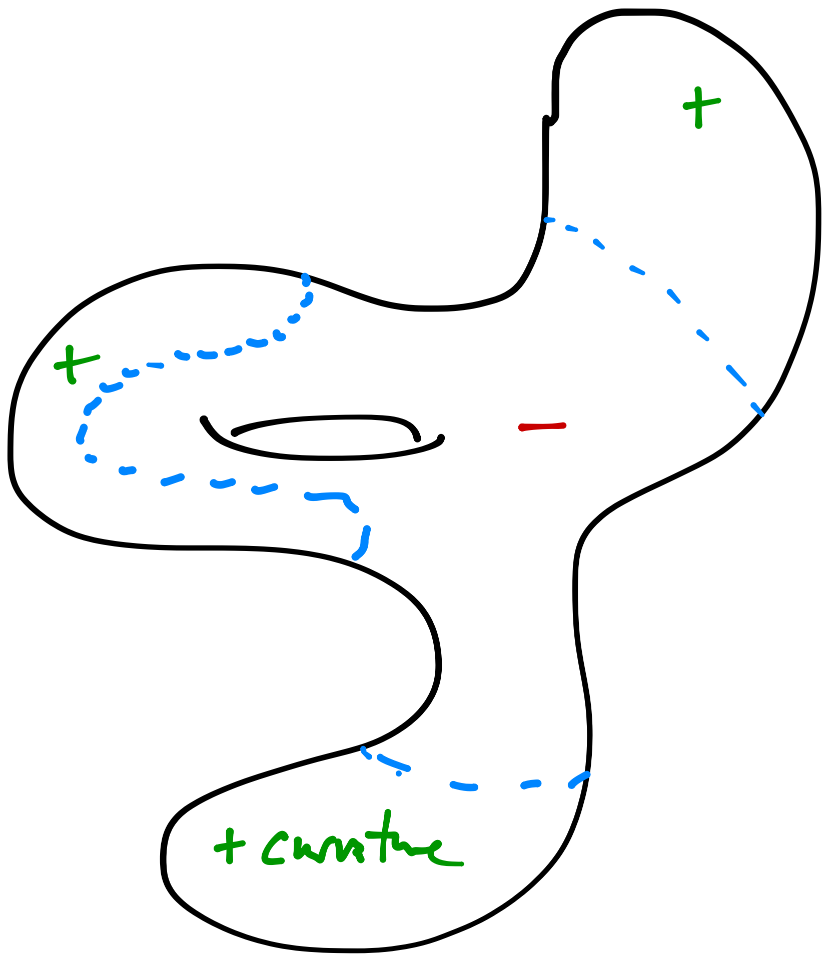


More negative curvature.

And more amazing global theorems:

Gauss-Bonnet theorem: The total curvature of any torus is 0.





And we'll even see an explicit example of how curvature arises in relativity.