Math 474 HW #6

Due 2:00 PM Friday, Dec. 11

1. The goal of this problem is to determine which surfaces of revolution have constant Gaussian curvature. Suppose the surface of revolution Σ has parametrization

$$\vec{x}(u,v) = (\phi(v)\cos u, \phi(v)\sin u, \psi(v)),$$

where the template curve $\alpha(s) = (\phi(s), 0, \psi(s))$ is parameterized by arclength (which of course means that $(\phi')^2 + (\psi')^2 = 1$). The goal is to solve for ϕ and ψ so that Σ has constant Gaussian curvature K.

(a) Prove that ϕ and ψ satisfy

$$\phi''(v) + K\phi(v) = 0 \quad \text{and} \quad \psi(v) = \int_0^v \sqrt{1 - (\phi'(t))^2}\,dt.$$

(b) Now assume $K = 1$ and show that, assuming the initial condition $\phi'(0) = 0$, the solutions of the equations from (a) are

$$\phi(v) = C\cos v \quad \text{and} \quad \psi(v) = \int_0^v \sqrt{1 - C^2\sin^2 t}\,dt,$$

where C is a constant. Obviously, $\psi(v)$ is not defined for all v; find the domain of $\psi(v)$ (which depends on C) and sketch the curve $\alpha(s) = (\phi(s), 0, \psi(s))$ for $C < 1$, $C = 1$, and $C > 1$ (feel free to use Mathematica, Maple, Matlab, Wolfram Alpha, etc. for this). Show that only the $C = 1$ surface can be rotated around the z-axis to get a compact regular surface.

(c) Now assume $K = -1$ and show that ϕ and ψ satisfy one of the following set of equations:

$$\phi(v) = C\cosh v \quad \text{and} \quad \psi(v) = \int_0^v \sqrt{1 - C^2\sinh^2 t}\,dt \tag{1}$$

$$\phi(v) = C\sinh v \quad \text{and} \quad \psi(v) = \int_0^v \sqrt{1 - C^2\cosh^2 t}\,dt \tag{2}$$

$$\phi(v) = e^v \quad \text{and} \quad \psi(v) = \int_0^v \sqrt{1 - e^{2t}}\,dt. \tag{3}$$

In each case, determine the domain of $\psi(v)$ and sketch the resulting surface.

(d) Finally, assume $K = 0$. Prove that the only solutions are the cylinder, the cone, and the plane.

2. Show that if $F = 0$ then the Gaussian curvature K of a surface Σ is given by

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right].$$

3. Let Σ be an oriented regular surface and let $\alpha(s)$ be an arclength parametrized curve on Σ. Since α lies on Σ, we know that $\alpha'(s) = T(s) \in T_{\alpha(s)}\Sigma$, and in particular $T(s)$ is perpendicular to the surface normal $N_\Sigma(s)$.

The *Darboux frame* of α is defined to be the triple of vectors

$$(T(s), V(s) = N_\Sigma(s) \times T(s), N_\Sigma(s)).$$

Like the Frenet frame, this frame's derivatives give information about the local geometry of α, but now that information relates also to how α lies in Σ.

(a) Show that the Darboux frame satisfies a system of equations vaguely similar to the Frenet equations:

$$\begin{aligned} T' &= \qquad\quad a(s)V(s) + b(s)N_\Sigma(s) \\ V' &= -a(s)T \qquad\qquad + c(s)N_\Sigma(s) \\ N_\Sigma' &= -b(s)T - c(s)V(s) \end{aligned}$$

for some coefficient functions $a(s)$, $b(s)$, and $c(s)$, which interpret in the following parts.

(b) Show that $c(s) = -\langle N_\Sigma', V\rangle$. In particular, α is a line of curvature if and only if $c(s) = 0$. The function $-c(s)$ is called the *geodesic torsion* for obvious reasons.

(c) Show that $b(s)$ is the normal curvature κ_n of α.

(d) Show that $a(s)$ is the geodesic curvature κ_g of α.

4. Let Σ be the hyperboloid of revolution

$$\vec{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, \sinh v),$$

which can also be described implicitly by the equation $x^2 + y^2 - z^2 = 1$. Suppose $\alpha(s)$ is a geodesic on Σ which makes the angle $\phi(s)$ with the $\vec{x}_u$ direction at the point $\alpha(s) = \vec{x}(u(s), v(s))$ and that the angle $\phi(s)$ satisfies

$$\cos(\phi(s)) = \frac{1}{\cosh(v(s))}.$$

Show that the geodesic α spirals asymptotically into the circle $v = 0$.