Math 474 HW #5

Due 2:00 PM Friday, Nov. 13

Choose 5 of the following 8 problems to write up and turn in.

1. Let p be a hyperbolic point on a regular surface Σ, and let $\vec{v}_1, \vec{v}_2 \in T_p\Sigma$ be the principal directions at p. Show that the directions $\vec{v}_1$ and $\vec{v}_2$ bisect the asymptotic directions at p.

2. Let $\alpha(s)$ be a regular curve on the regular surface Σ, and let $\alpha(s_0)$ be a point on α. Suppose that Σ has Gaussian curvature $K > 0$ and principal curvatures k_1 and k_2 at $\alpha(s_0)$. Prove that the curvature $\kappa(s)$ of α satisfies
$$\kappa(s_0) \geq \min\{|k_1|, |k_2|\}.$$

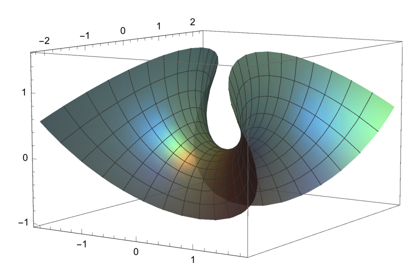
3. Let Σ be a regular surface with principal curvatures k_1 and k_2 with $|k_1| \leq 1$ and $|k_2| \leq 1$ at all points on Σ. Let α be a regular curve on Σ and let κ be its curvature. Is it true that $|\kappa| \leq 1$? Either prove this statement or give a specific counterexample.

4. Let $\alpha(s)$ be an asymptotic curve on a surface Σ with Gaussian curvature K so that the curvature of α is never zero. Prove that the torsion $\tau(s)$ of α satisfies
$$|\tau(s)| = \sqrt{-K}$$
(note that $K \leq 0$ at all points α passes through because otherwise there wouldn't be any asymptotic directions at those points.)

5. Suppose that $\vec{w}_1, \vec{w}_2$ are perpendicular directions in the tangent plane $T_p\Sigma$ to a regular surface Σ. Show that the sum $\kappa_n(\vec{w}_1) + \kappa_n(\vec{w}_2)$ does not depend on the specific choice of $\vec{w}_1$ and $\vec{w}_2$ so long as they are perpendicular.

6. Find the asymptotic curves and lines of curvature of the surface $\vec{x}(u, v) = (u, v, uv)$.

7. Consider Enneper's surface
$$\vec{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, -v + \frac{v^3}{3} - vu^2, u^2 - v^2\right).$$



Show that

(a) The first fundamental form is given by
$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

(b) The second fundamental form is given by

$$e = -2, \quad g = 2, \quad f = 0.$$

(c) The principal curvatures are

$$k_1 = \frac{2}{(1+u^2+v^2)^2}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}.$$

(d) Compute the Gaussian and mean curvatures for Enneper's surface.

8. (Shifrin Problem 2.2.13) Suppose Σ is a regular surface with Gaussian curvature $K > 0$. Is it true that the space curvature of any curve α on Σ is positive everywhere? Either prove this statement or give a specific counterexample.